NADA, KTH, Lennart Edsberg

Tentamen i Kursen 2D1225
Numerisk Behandling av Differentialekvationer I
Saturday 2006-12-16 kl 8–13

SOLUTIONS

1. See the compendium, chapter 2, pg 11. Eigenvalues of the first matrix are $\lambda_{1,2} = \pm 3i$, i.e. $Re(\lambda_i) = 0$ but they are simple, hence the system is *stable* (but not asymptotically stable). Eigenvalues of the second matrix are $\lambda_{1,2} = 0$, i.e. a double eigenvalues with $Re(\lambda_i) = 0$. We then have to investigate the particular system which is

$$\dot{u}_1 = 9u_2, \qquad \dot{u}_2 = 0$$

which gives $u_2 = C, \quad u_1 = 9Ct + D$, hence an *unstable* system.

2. The ansatz $y'(a) = Ay(a + h/2) + By(a + h) + Cy(a + 2h)$ leads after Taylor expansion to $y'(a) = (A + B + C)y(a) + h(A/2 + B + 2C)y'(a) + h^2(A/8 + B/2 + 2C)y''(a) + O(h^3)$, hence the linear system of equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 1/2 & 1 & 2 \\ 1/8 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 1/h \\ 0 \end{pmatrix}$$

and the solution $A = -4/h$, $B = 5/h$ and $C = -1/h$, hence

$$y'(a) = \frac{-4y(a + h/2) + 5y(a + h) - y(a + 2h)}{h} + O(h^2)$$

The error term comes from the first neglected term in the Taylor expansion:

$$(-\frac{4}{h}\frac{h^3}{48} + \frac{5}{h}\frac{h^3}{6} - \frac{1}{h}\frac{4h^3}{3})y'''(a) = -\frac{7}{12}h^2 y'''(a) = O(h^2)$$

3. The stability region of Euler's explicit method, see the compendium chapter 3, pg 13. - Inserting the right hand side of $\dot{y} = qy$ into the RK-formula gives $y_{k+1} = (1 + hq + (hq)^2/2)y_k$, hence the stability area in the hq-plane is defined by $|1 + hq + (hq)^2/2| \leq 1$.

4. Se the compendium, chapter 7, pg 12.

5. When using linear triangle elements the FEM-solution is piecewise linear, which means that the FEM-solution $\tilde{u}$ restricted to a triangle T is linear, i.e. $\tilde{u} = a + bx + cy$. Inserting the three points gives the following three equations: $a + b - 2c = 2$, $a + b + c = 1$, $a - 2b + c = 3$ which gives $\tilde{u}(0, 0) = a = 2$

6. The curves are called characteristics. The characteristics of the advection equation satisfy the straight lines $x = at + C$. If c depends on t we have $dx/dt = c(t) = c_0 + c_1 t$, hence $x = c_0 t + c_1 t^2/2 + D$.

7. The solution should repeat itself periodically. For a 2nd order ODE, the BCs could be $u(a) = u(b)$, $u'(a) = u'(b)$.

8. Let $T = T_{out} + (T_{init} - T_{out})u$, $r = Rx$, $t = \alpha\tau$, where α is to be conveniently chosen later. Insert into the PDE:

$$\frac{\partial(T_{out} + (T_{init} - T_{out})u)}{\partial(\alpha\tau)} = \kappa\frac{1}{(Rx)^2}\frac{\partial}{\partial(Rx)}((Rx)^2\frac{\partial(T_{out} + (T_{init} - T_{out})u)}{\partial(Rx)}$$

and we get after some algebraic manipulations:

$$\frac{\partial u}{\partial \tau} = \frac{\alpha\kappa}{R^2}\frac{1}{x^2}\frac{\partial}{\partial x}(x^2\frac{\partial u}{\partial x})$$

Now, choose $\alpha = R^2/\kappa$ having dimension $[m^2/(m^2/s)] = [s]$ and the required PDE is derived. The initial condition:

$$T(r,0) = T_{init} = T_{out} + (T_{init} - T_{out})u(x,0) \rightarrow u(x,0) = 1$$

The boundary conditions:

$$\frac{\partial T}{\partial r}(0,t) = 0 \rightarrow \frac{\partial u}{\partial x}(0,\tau) = 0$$

$$k\frac{\partial(T_{out} + (T_{init} - T_{out})u)}{\partial(Rx)} = -\beta(T_{init} - T_{out})u \rightarrow \frac{\partial u}{\partial x}(1,\tau) = au(1,\tau)$$

where $a = R\beta/\kappa$ and the dimension of a is $[(m \cdot J \cdot m \cdot K)/(m^2 \cdot K \cdot J)] = 1$.

10a) $x_i = ih, i = -1, 0, \ldots, N, N+1$, where $h = 1/N$, i.e. $x_0 = 0, x_N = 1$

10b) Rewrite first the PDE:

$$\frac{\partial u}{\partial \tau} = \frac{2}{x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}$$

The MoL gives the ODE-system where the space derivatives are approximated with 2nd order accuracy:

$$\frac{du_i}{d\tau} = \frac{2}{x_i}\frac{u_{i+1} - u_{i-1}}{2h} + \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, i = 0, 1, 2, \ldots, N$$

This MoL-approximation does not work for $i = 0$, where $x_i = 0$. At that point we have to investigate what happens to the right hand side of the ODE using l'Hopital's rule:

$$lim_{x->0}(\frac{2u'(x)}{x} + u''(x)) = 3u''(0)$$

Hence at $x = 0$, the MoL-discretized PDE is

$$\frac{du_0}{d\tau} = 3\frac{u_1 - 2u_0 + u_{-1}}{h^2}$$

10c) With centraldifference approximations we obtain:

$$\frac{u_1 - u_{-1}}{2h} = 0, \qquad \frac{u_{N+1} - u_{N-1}}{2h} = au_N$$

10d) Eliminating u_{-1} and u_{N+1} we obtain

$$\frac{du_0}{d\tau} = 3\frac{2u_1 - 2u_0}{h^2}, \quad u_0(0) = 1$$

$$\frac{du_i}{d\tau} = \frac{2}{x_i}\frac{u_{i+1} - u_{i-1}}{2h} + \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \quad u_i(0) = 1, \quad i = 1, 2, \ldots, N-1$$

$$\frac{du_N}{d\tau} = 2au_N + \frac{((-2 + 2ha)u_N + 2u_{N-1})}{h^2}, \quad u_N(0) = 1$$

10e) Since $\partial u/\partial \tau = 0$, we obtain

$$\frac{1}{x^2}\frac{d}{dx}(x^2\frac{du}{dx}) = 0 \rightarrow x^2\frac{du}{dx} = C \rightarrow u = -\frac{C}{x} + D$$

The constant $C = 0$, since the solution should exist at $x = 0$, hence $u = D$. From the boundary condition at $x = 1$ we obtain $u = 0$, which is not unexpected, since this corresponds to $T = T_{out}$, which means that the sphere takes the same temperature as the environment, when it has cooled off.