NADA, KTH, Lennart Edsberg

Tentamen i Kursen 2D1225
Numerisk Behandling av Differentialekvationer I
Thursday 2008-12-18 kl 8–13

SOLUTIONS

1. a) The definition of e^{At} involving eigenvalues and eigenvectors is

$$e^{At} = Se^{Dt}S^{-1}$$

where $e^{Dt} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$

b) The eigenvalues of A are -1 and -2 (for a triangular matrix the eigenvalues are found in the diagonal of A). The homogeneous linear systems of equations giving the eigenvectors are

$$\lambda_1 = -1 \quad \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \quad \lambda_2 = -2 \quad \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

giving $\mathbf{x} = (1, 2)^T$ and $\mathbf{y} = (0, 1)^T$. We obtain

$$S = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

c) If the columns of S are scaled by numbers $w_1, w_2, \ldots, w_n$ we get $S_w = S{*}W$, where W is a diagonal matrix with $w_1, w_2, \ldots w_n$ in the diagonal, hence $S = S_w * W^{-1}$. The inverse is $S^{-1} = W * S_w^{-1}$. Inserting into the definition gives $e^{At} = S_w * W^{-1} * e^{Dt} * W * S_w^{-1}$. The three matrices in the middle, $W^{-1} * e^{Dt} * W$ are all diagonal matrices, and the order of multiplication can be changed to e.g. $W^{-1} * W * e^{Dt} = e^{Dt}$. Hence e^{At} does not depend on the lengths of the eigenvectors.
Another argument for this is to look at the Taylor expansion definition of $e^{At} = I + At + (At)^2/2! + (At)^3/3! + ...$, which is independent of S.

2. a) Method 1: Solve the differential equation exactly.
Let $v = u', \rightarrow v' + \lambda v = 0$. If $\lambda \neq 0$ we get $v = Ce^{-\lambda x} \rightarrow u = C_1 + C_2 e^{-\lambda x}$. The solution is stable if $Re(\lambda) \geq 0$. If $\lambda = 0$, however, we have $u'' = 0 \rightarrow u = C_1 + C_2 x$ which is unstable.
Method 2: Write as a system of two first order ODE's where $u_1 = u, u_2 = u'$:

$$\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 0 & -\lambda \end{pmatrix} \mathbf{u}$$

The eigenvalues are 0 and $-\lambda$. If $\lambda \neq 0$ we have two simple eigenvalues and the system is stable if $Re(\lambda) \geq 0$. If $\lambda = 0$, both eigenvalues $= 0$ and we have a double eigenvalue. To investigate stability in this case we have to look at the solution of the differential equation, i.e. $u'' = 0$, which is done in method 1, showing that it is unstable.

b) For the differential equation: $\mu^2 + \lambda\mu = 0 \qquad (1)$. For the difference equation: $(1 + h\lambda)\nu^2 - (2 + h\lambda)\nu + 1 = 0 \qquad (2)$

c) The solution of the difference equation is stable if $|\nu_1| \leq 1$ and $|\nu_2| \leq 1$. The roots of (2) are $\nu_1 = 1$ and $\nu_2 = 1/(1 + h\lambda)$. If $\lambda \neq 0$ we have two simple roots and a stable solution if $|1 + h\lambda| > 1$, i.e. $h\lambda$ is in the outside of a disc with radius 1 and center at -1. If $\lambda = 0$ we have a double root. The difference equation is then: $u_n - 2u_{n-1} + u_{n-2} = 0$ and the solution $u_n = C_1 + C_2 n$, hence unstable.

3. Taylor expansion of $ay(-h) + by(0) + cy(h) + dy(2h)$ gives $(a + b + c + d)y(0) + h(-a + c + 2d)y'(0) + (h^2/2!)(a + c + 4d)y''(0) + (h^3/3!)(-a - c + 8d)y'''(0) + (h^4/4!)(a + c + 16d)y(4)(0) + O(h^5)$, hence the linear system of equations is obtained by identifying $y'''(0)$ with the first four terms in the Taylor expansion:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6/h^3 \end{pmatrix}$$

and the solution $a = -1/h^3$, $b = 3/h^3$, $c = -3/h^3$ and $d = 1/h^3$, hence

$$y'''(0) = \frac{-y(-h) + 3y(0) - 3y(h) + y(2h)}{h^3} + \frac{h}{2}y^4(0) + ...$$

and the approximation is of first order.

4. See the book, pg 156. The characteristics are the parallell straight lines $x = 2t + C$.

5. See the book chapter 6.5 and pg 114-115

6. a) The PDE is parabolic

b) Discretize the r-axis according to $r_0 = -h, r_1 = 0, r_2 = h, r_i = (i-1)h, r_{N+1} = R$, i.e. $h = R/N$. Use the MoL to obtain a system of N ODEs at $r = r_1, r_2, \dots r_N$ with z as independent variable. At $r = r_1$ we have to investigate the term $(1/r)\partial T/\partial r$, which is of type 0/0. Use of l'Hôpital's rule gives

$$lim_{r->0}\frac{\partial T/\partial r}{r} = lim_{r->0}\frac{\partial^2 T/\partial r^2}{1} = \frac{\partial^2 T}{\partial r^2}(0, z)$$

Hence we get the following system of ODEs

$$at \quad r = r_1 = 0: \qquad \frac{dT_1}{dz} = 2\frac{D_r}{v}\frac{T_2 - 2T_1 + T_0}{h^2}$$

$$at \quad r = r_i: \qquad \frac{dT_i}{dz} = \frac{D_r}{v}(\frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} + \frac{1}{r_i}\frac{T_{i+1} - T_{i-1}}{2h}), i = 2, 3, \dots, N$$

The BC's are discretized according to

$$at \quad r = r_1 = 0: \quad \frac{T_2 - T_0}{2h} = 0, \qquad at \quad r = r_{N+1} = R: T_{N+1} = T_{out}$$

giving an ODE-system for $\mathbf{T} = (T_1, T_2, \dots, T_N)^T$:

$$\frac{d\mathbf{T}}{dz} = A\mathbf{T} + \mathbf{b}, \quad \mathbf{T}(0) = \mathbf{T}_0$$

where

$$A = \begin{pmatrix} -4\alpha & 4\alpha & 0 & 0 & 0 \\ \alpha - \beta_2 & -2\alpha & \alpha + \beta_2 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \alpha - \beta_N & -2\alpha \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ -(\alpha + \beta_N)T_{out} \end{pmatrix}, \quad \mathbf{T}(0) = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

where $\alpha = D_r/vh^2$ and $\beta_i = D_r/2hvr_i$. Hence A is tridiagonal with the elements $\alpha - \beta_i, -2\alpha, \alpha + \beta_i$ in row i.

c) The PDE is elliptic. You need a boundary condition also at $z = L$, i.e. $T(r, L) = T_L$ or $\partial T/\partial z = 0$. The FEM or the FDM can be used to discretize the problem. A large sparse linear system of equations is obtained. Can be solved with a sparse algorithm or iteration with CG or preconditioned CG.