## Homework 1, DN2230

Due November 2, 2012

If a correct solution is handed in before the deadline (i.e. November 2) two bonus points will be awarded to the final written exam. If a solution that is handed in before that date is not correct, it has to be redone, but the second time without yielding bonus points for the exam.

Solutions must be clearly written, and easy to follow. If not, they will not generate bonus points, and must be redone.

Together with the written solution, I want you to send me your code by e-mail.

This homework is to implement the multigrid method for the simple one-dimensional Poisson equation:

$$-u''(x) = f(x), \quad \text{in } (0,1), \quad u(0) = u(1) = 0.$$

Discretize the equation using finite differences. Denote by $\Omega^h$ the discrete solution space corresponding to the step length $h$. For simplicity we let the different grids be given by step lengths $2^{-i}$, for different integers $i$. Use linear interpolation for the prolongation operator $p : \Omega^{2h} \to \Omega^h$. Written in matrix form this gives

$$p(v^{2h}) = \frac{1}{2} \begin{pmatrix} 1 & & \\ 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & 1 \\ & & 2 \\ & & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = v^h.$$

The restriction operator $r : \Omega^h \to \Omega^{2h}$ is defined by weighting:

$$r(v^h) = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v^{2h}.$$

(Here we used the spaces with step lengths $h = 1/8$ and $2h = 1/4$.)

You may implement the smoothing iterations using damped Jacobi,

$$v^{n+1} = [(1-\omega)I + \omega D^{-1}(L+U)]v^n + \omega D^{-1}b,$$

for the linear system $Av = b$, where $A = D - L - U$ ($D$, $-L$, and $-U$ are the diagonal, lower, and upper parts of $A$). As we saw in the lecture the damping $\omega = 2/3$ gives fast reduction of the highly oscillatory modes.

## PROBLEMS

Implement the multi-grid V- and W-cycles. Compare their performances in plots of the convergence history of the solution $\|v - v^{\text{true}}\|$ (where $v^{\text{true}}$ is the true discrete solution) and the residual $\|b - Av\|$ as a function of the iteration number. Use the same number of pre-smoothing steps, $\nu_1$, and post-smoothing steps, $\nu_2$, for both the V- and W-cycle.

Also compare the performance of the one-grid (only damped Jacobi iteration), two-grid and three-grid iterations. Instead of solving the equation on the coarsest grid exactly, use only one or two smoothing iterations there. How much more efficient is two-, three-, and multi-grid compared to single grid iteration?

Let the initial distribution $v^0$ have both a smooth and an oscillatory part. Choose your own favourite non-zero right hand side function $f$.