Homework 3, DN2230

Due November 27, 2012

If a correct solution is handed in before the deadline (i.e. November 27) two bonus points will be awarded to the final written exam. If a solution that is handed in before that date is not correct, it has to be redone, but the second time without yielding bonus points for the exam. Question 2 is optional, but if not solved before the deadline, it is not possible to get two bonus points on this homework.

Solutions must be clearly written, and easy to follow. If not, they will not generate bonus points, and must be redone.

Together with your solution, I want you to send me your code (for exercise 3) by e-mail.

1. Exercise 28.4 (b) in NLA. Use your result from question 2 here. We are again interested in the orders of the methods, i.e. to get exponents α as in question 3.

2. (Optional, but generating bonus) Exercise 28.4 (a) in NLA. For the bonus point you must present an algorithm that gives the optimal computational work order.

3. Exercise 29.1 in NLA. In addition to the task in part (e), try your program to see how large a matrix $A \in \mathbb{R}^{m \times m}$ could be such that your program is able to find all the eigenvalues in a reasonable amount of time (say 30 seconds). Is it different when you use the unshifted QR algorithm and when you use the Wilkinson shift? Use matrices A similar to the one in the exercise (i.e. `A=diag(m:-1:1)+ones(m,m)`).

4. The goal of this exercise is to prove the following theorem.

   Theorem. *Let A be any non-singular $m \times m$ matrix and b any vector of length m. The GMRES method finds the exact solution of $Ax = b$ in at most m steps (i.e. $r_n = 0$ for some $n \leq m$).*

   You will construct the proof by solving each of the following subproblems:

   (a) Show that if $h_{n+1,n} \neq 0$ for $n \leq m-1$ in the Arnoldi iteration (no Arnoldi breakdown), then $\mathcal{K}_m = \mathbb{C}^m$.

(b) Under the assumption in (a), show that the m-th iterate of the GMRES method satisfies the equation.

(c) Assume that at some n, $h_{n+1,n} = 0$ in the Arnoldi iteration (Arnoldi breakdown). Show that $\mathcal{K}_n$ is an invariant subspace of A, i.e. $Av \in \mathcal{K}_n$, for every $v \in \mathcal{K}_n$.

(d) Under the assumption in (c), show that $AQ_n = Q_n H_n$, where Q_n and H_n are the matrices defined in equations (33.1) and (33.8) in NLA.

(e) Under the assumption in (c), show that H_n is invertible. This may for instance be seen by proving that each eigenvalue of H_n is an eigenvalue of A. Hence 0 can not be an eigenvalue, so H_n must be invertible.

(f) Under the assumption in (c), show that the solution x to the system of equations $Ax = b$ lies in $\mathcal{K}_n$. Conclude that GMRES has found the solution to $Ax = b$ in step n.

5. Exercise 38.5 in "Numerical Linear Algebra" (NLA).

Good luck!