Chapter 6

Finite Difference Methods

This section introduces finite difference methods for approximation of partial differential equations. We first apply the finite difference method to a partial differential equation for a financial option problem, which is more efficiently computed by partial differential methods than Monte Carlo techniques. Then we discuss the fundamental Lax Equivalence Theorem, which gives the basic understanding of accuracy and stability for approximation of differential equations.

6.1 American Options

Assume that the stock value, S(t), evolves in the risk neutral formulation by the Itô geometric Brownian motion

$$dS = rSdt + \sigma SdW.$$

An American put option is a contract that gives the possibility to sell a stock for a fixed price K up to time T. Therefore the derivation of option values in Chapter 4 shows that European and American options have the formulations:

1. The price of an European put option is

$$f(t,s) \equiv E[\ e^{-r(T-t)} \max\big(K - S(T), 0\big) |\ S(t) = s\ ].$$

2. The price of an American option is obtained by maximizing over all sell time τ strategies, which depend on the stock price up to the sell

time,

$$f_A(t,s) \equiv \max_{t\le\tau\le T} E[\; e^{-r(\tau-t)} \max\big(K - S(\tau),0\big)|\; S(t) = s\;]. \qquad (6.1)$$

How to find the optimal selling strategy for an American option? Assume that selling is only allowed at the discrete time levels $0, \Delta t, 2\Delta t, \dots, T$. Consider the small time step $(T - \Delta t, T)$. By assumption the option is not sold in the step. Therefore the European value $f(t,s)$ holds, where $f(T,s) = \max(K - s, 0)$ and for $T - \Delta t < t < T$

$$f_t + rSf_S + \frac{1}{2}\sigma^2 S^2 f_{SS} = rf. \qquad (6.2)$$

If, for a fixed stock price $s = S(T-\Delta t)$, there holds $f(T-\Delta t, s) < \max(K - s, 0)$ then keeping the option gives the expected value $f(T - \Delta t, s)$ which is clearly less than the value $\max(K - s, 0)$ obtained by selling at time $T - \Delta t$. Therefore it is optimal to sell if $f(T - \Delta t, s) < \max(K - s, 0) \equiv f_F$. Modify the initial data at $t = T - \Delta t$ to $\max(f(T - \Delta t, s), f_F)$ and repeat the step (6.2) for $(T - 2\Delta t, T - \Delta t)$ and so on. The price of the American option is obtained as the limit of this solution as $\Delta t \to 0$.

Example 6.1 A corresponding Monte Carlo method based on (6.1) requires simulation of expected values $E[e^{-r\tau}\max(K - S(\tau), 0)]$ for many different possible selling time strategies τ until an approximation of the maximum values is found. Since the τ need to depend on ω, with M time steps and N realizations there are M^N different strategies.

Note that the optimal selling strategy

$$\tau = \tau^* = \inf_v \{v : t \le v \le T, f_A\big(v, S(v)\big) = \max\big(K - S(v), 0\big)\}$$

for the American option, which is a function of f_A, seems expensive to evaluate by Monte Carlo technique, but is obtained directly in the partial differential formulation above and below. This technique is a special case of the so called dynamic programming method, which we shall study systematically for general optimization problems in a later Chapter, cf. also the last example in Chapter 1.

Here and in Exercise 6.2 is a numerical method to determine the value of an American option:

(1) Discretize the computational domain $[0,T] \times [s_0, s_1]$ and let

$$f_A(n\Delta t, i\Delta S) \simeq \bar{f}_{n,i}, \quad \bar{f}_{N,i} = \max\big(K - i\Delta S, 0\big).$$

(2) Use the Euler and central difference methods for the equation (6.2)

$$\begin{array}{ll} \partial_t f_A \simeq \frac{\bar{f}_{n,i} - \hat{f}_{n-1,i}}{\Delta t} & \partial_S f_A \simeq \frac{\bar{f}_{n,i+1} - \bar{f}_{n,i-1}}{2\Delta S} \\ \partial_{SS} f_A \simeq \frac{\bar{f}_{n,i+1} - 2\bar{f}_{n,i} + \bar{f}_{n,i-1}}{(\Delta S)^2} & f_A \simeq \bar{f}_{n,i}. \end{array}$$

(3) Make a Black-Scholes prediction for each time step

$$\begin{aligned} \hat{f}_{n-1,i} &= \bar{f}_{n,i}(1 - r\Delta t - \sigma^2 i^2 \Delta t) + \bar{f}_{n,i+1}(\frac{1}{2} ri\Delta t + \frac{1}{2}\sigma^2 i^2 \Delta t) \\ &+ \bar{f}_{n,i-1}(-\frac{1}{2} ri\Delta t + \frac{1}{2}\sigma^2 i^2 \Delta t). \end{aligned}$$

(4) Compare the prediction with selling by letting

$$\bar{f}_{n-1,i} = \max\big(\hat{f}_{n-1,i}, \max(K - i\Delta S, 0)\big),$$

and go to the next time Step 3 by decreasing n by 1.

Exercise 6.2 The method above needs in addition boundary conditions at $S = s_0$ and $S = s_1$ for $t < T$. How can s_0, s_1 and these conditions be choosen to yield a good approximation?

Exercise 6.3 Give a trinomial tree interpretation of the finite difference scheme

$$\begin{aligned} \bar{f}_{n+1,i} &= \bar{f}_{n,i}(1 + r\Delta t + \sigma^2 i^2 \Delta t) + \bar{f}_{n,i+1}(-\frac{1}{2} ri\Delta t - \frac{1}{2}\sigma^2 i^2 \Delta t) \\ &+ \bar{f}_{n,i-1}(\frac{1}{2} ri\Delta t - \frac{1}{2}\sigma^2 i^2 \Delta t), \end{aligned}$$

for Black-Scholes equation of an European option. Binomial and trinomial tree approximations are frequent in the finance economy literature, cf. [J. Hull].

Let us now study general finite difference methods for partial differential equations. The motivation to introduce general finite difference methods in contrast to study only the binomial and trinomial tree methods is that higher order methods, such as the Crank-Nicolson method below, are more efficient to solve e.g. (6.2).

The error for the binomial and the trinomial tree method applied to the partial differential equation (6.2) for a European option is $\varepsilon = \mathcal{O}(\Delta t + (\Delta s)^2)$, which is clearly the same for the related forward and backward Euler methods. The work is then $\mathcal{A} = \mathcal{O}((\Delta t \Delta s)^{-1})$, so that $\mathcal{A} = \mathcal{O}(\varepsilon^{-3/2})$. For the Crank-Nicolsen method the accuracy is $\varepsilon = \mathcal{O}((\Delta t)^2 + (\Delta s)^2)$ and the work is still $\mathcal{A} = \mathcal{O}((\Delta t \Delta s)^{-1})$, which implies the improved bound $\mathcal{A} = \mathcal{O}(\varepsilon^{-1})$. For a general implicit method with a smooth exact solution in $[0,T] \times \mathbb{R}^d$ the accuracy is $\varepsilon = \mathcal{O}((\Delta t)^q + (\Delta s)^p)$ with the miminal work (using e.g. the multigrid method) $\mathcal{A} = \mathcal{O}(\frac{q^2}{\Delta t}(\frac{p^2}{\Delta s})^d)$, which gives $\mathcal{A} = \mathcal{O}(\frac{q^2}{\varepsilon^{1/q}}(\frac{p^2}{\varepsilon^{1/p}})^d)$. In the next section we derive these error estimates for some model problems.

6.2 Lax Equivalence Theorem

Lax equivalence theorem defines the basic concepts for approximation of linear well posed differential equations. Here, well posed means that the equation is solvable for data in a suitable function space and that the solution operator is bounded. We will first formally state the result without being mathematically precise with function spaces and norms. Then we present two examples with proofs based on norms and functions spaces.

The ingredients of Lax Equivalence Theorem 6.4 are:

(0) an exact solution u, satisfying the *linear well posed equation* $Lu = f$, and an approximation u_h, obtained from $L_h u_h = f_h$;

(1) *stability*, the approximate solution operators $\|L_h^{-1}\|$ are uniformly bounded in h and the exact solution operator $\|L^{-1}\|$ is bounded;

(2) *consistency*, $f_h \to f$ and $L_h u \to Lu$ as the mesh size $h \to 0$; and

(3) *convergence*, $u_h \to u$ as the mesh size $h \to 0$.

Theorem 6.4 *The combination of stability and consistency is equivalent to convergence.*

The idea of the proof. To verify convergence, consider the identity

$$u - u_h = L_h^{-1}[\ L_h u - L_h u_h\ ] \stackrel{Step(0)}{=} L_h^{-1}[\ (L_h u - Lu) + (f - f_h)\ ].$$

Stability implies that L_h^{-1} is bounded and consistency implies that

$$L_h u - Lu \to 0 \text{ and } f - f_h \to 0,$$

and consequently the convergence holds

$$\begin{aligned}\lim_{h\to 0}(u - u_h) &= \lim_{h\to 0} L_h^{-1}[\ (L_h u - Lu) + (f - f_h)\ ] \\ &= 0.\end{aligned}$$

Clearly, consistency is necessary for convergence. Example 6.7, below, indicates that also stability is necessary. □

Let us now more precisely consider the requirements and norms to verify stability and consistency for two concrete examples of ordinary and partial differential equations.

Example 6.5 Consider the forward Euler method for the ordinary differential equation

$$\begin{aligned}&u'(t) = Au(t) \quad 0 < t < 1,\\ &u(0) = u_0.\end{aligned} \tag{6.3}$$

Verify the conditions of stability and consistency in Lax Equivalence Theorem.

Solution. For a given partition, $0 = t_0 < t_1 < ... < t_N = 1$, with $\Delta t = t_{n+1} - t_n$, let

$$\begin{aligned}u_{n+1} &\equiv (I + \Delta t A)u_n \\ &= G^n u_0 \qquad \text{where } G = (I + \Delta t A).\end{aligned}$$

Then:

(1) Stability means $|G^n| + |H^n| \le e^{Kn\Delta t}$ for some K, where $|\cdot|$ denotes the matrix norm $|F| \equiv \sup_{\{v\in\mathbb{R}^n:|v|\le 1\}} |Fv|$ with the Euclidean norm $|w| \equiv \sqrt{\sum_i w_i^2}$ in $\mathbb{R}^n$.

(2) Consistency means $|(G-H)v| \leq C(\Delta t)^{p+1}$, where $H = e^{\Delta t A}$ and p is the order of accuracy. In other words, the consistency error $(G-H)v$ is the local approximation error after one time step with the same initial data v.

This stability and consistency imply the convergence

$$\begin{aligned} \mid u_n - u(n\Delta t) \mid &= \mid (G^n - H^n)u_0 \mid \\ &= \mid (G^{n-1} + G^{n-2}H + ... + GH^{n-2} + H^{n-1})(G-H)u_0 \mid \\ &\leq \mid G^{n-1} + G^{n-2}H + ... + GH^{n-2} + H^{n-1}||(G-H)u_0 \mid \\ &\leq C(\Delta t)^{p+1} n \mid u_0 \mid e^{Kn\Delta t} \\ &\leq C'(\Delta t)^p, \end{aligned}$$

with the convergence rate $\mathcal{O}(\Delta t^p)$. For example, $p = 1$ in case of the Euler method and $p = 2$ in case of the trapezoidal method. □

Example 6.6 Consider the heat equation

$$\begin{aligned} u_t &= u_{xx} \quad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{6.4}$$

Verify the stability and consistency conditions in Lax Equivalence Theorem.

Solution. Apply the Fourier transform to equation (6.4),

$$\hat{u}_t = -\omega^2 \hat{u}$$

so that

$$\hat{u}(t, \omega) = e^{-t\omega^2} \hat{u}_0(\omega).$$

Therefore $\hat{H} = e^{-\Delta t \omega^2}$ is the exact solution operator for one time step, i.e. $\hat{u}(t + \Delta t) = \hat{H}\hat{u}(t)$. Consider the difference approximation of (6.4)

$$\frac{u_{n+1,i} - u_{n,i}}{\Delta t} = \frac{u_{n,i+1} - 2u_{n,i} + u_{n,i-1}}{\Delta x^2},$$

which shows

$$u_{n+1,i} = u_{n,i}\left(1 - \frac{2\Delta t}{\Delta x^2}\right) + \frac{\Delta t}{\Delta x^2}\left(u_{n,i+1} + u_{n,i-1}\right),$$

where $u_{n,i} \simeq u(n\Delta t, i\Delta x)$. Apply the Fourier transform to obtain

$$\begin{aligned}
\hat{u}_{n+1} &= \left[\left(1-\frac{2\Delta t}{\Delta x^2}\right)+\frac{\Delta t}{\Delta x^2}\left(e^{j\Delta x\omega}+e^{-j\Delta x\omega}\right)\right]\hat{u}_n \\
&= \left[1-2\frac{\Delta t}{\Delta x^2}+2\frac{\Delta t}{\Delta x^2}\cos(\Delta x\omega)\right]\hat{u}_n \\
&= \hat{G}\hat{u}_n \qquad (\text{ Let } \hat{G} \equiv 1-2\frac{\Delta t}{\Delta x^2}+2\frac{\Delta t}{\Delta x^2}\cos(\Delta x\omega)) \\
&= \hat{G}^{n+1}\hat{u}_0.
\end{aligned}$$

1. We have

$$\begin{aligned}
2\pi\|u_n\|_{L^2}^2 &= \|\hat{u}_n\|_{L^2}^2 \qquad \text{(by Parseval's formula)} \\
&= \|\hat{G}^n\hat{u}_0\|_{L^2}^2 \\
&\leq \sup_\omega |\hat{G}^n|^2 \ \|\hat{u}_0\|_{L^2}^2.
\end{aligned}$$

   Therefore the condition

$$\|\hat{G}^n\|_{L^\infty} \leq e^{Kn\Delta t} \tag{6.5}$$

   implies L^2-stability.

2. We have

$$2\pi\|u_1 - u(\Delta t)\|_{L^2}^2 = \|\hat{G}\hat{u}_0 - \hat{H}\hat{u}_0\|_{L^2}^2,$$

   where u_1 is the approximate solution after one time step. Let $\lambda \equiv \frac{\Delta t}{\Delta x^2}$, then we obtain

$$\begin{aligned}
|(\hat{G}-\hat{H})\hat{u}_0| &= |\left(1-2\lambda+2\lambda\cos\Delta x\omega - e^{-\Delta t\omega^2}\right)\hat{u}_0| \\
&= \mathcal{O}(\Delta t^2)\omega^4|\hat{u}_0|,
\end{aligned}$$

   since for $0 \leq \Delta t\omega^2 \equiv x \leq 1$

$$\begin{aligned}
|1-2\lambda &+ 2\lambda\cos\sqrt{x/\lambda} - e^{-x}| \\
&= \left(1-2\lambda+2\lambda\left(1-\frac{x}{2\lambda}+\mathcal{O}(x^2)\right)-\left(1-x+\mathcal{O}(x^2)\right)\right) \\
&\leq Cx^2 = C(\Delta t)^2\omega^4,
\end{aligned}$$

   and for $1 < \Delta t\omega^2 = x$

$$|1-2\lambda+2\lambda\cos\sqrt{x/\lambda} - e^{-x}| \leq C = C\frac{(\Delta t)^2\omega^4}{x^2} \leq C(\Delta t)^2\omega^4.$$

Therefore the consistency condition reduces to

$$\begin{aligned} \| \; (\hat{G}-\hat{H})\hat{u}_0 \; \| &\leq \|K\Delta t^2 \omega^4 \hat{u}_0\| \\ &\leq K\Delta t^2 \|\partial_{xxxx} u_0\|_{L^2}. \end{aligned} \tag{6.6}$$

3. The stability (6.5) holds if

$$\|\hat{G}\|_{L^\infty} \equiv \sup_\omega |\hat{G}(\omega)| = \max_\omega |1-2\lambda+2\lambda\cos\Delta x\omega| \leq 1, \tag{6.7}$$

which requires

$$\lambda = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}. \tag{6.8}$$

The L^2-stability condition (6.7) is called the von Neuman stability condition.

4. Convergence follows by the estimates (6.6), (6.7) and $\|\hat{H}\|_{L^\infty} \leq 1$

$$\begin{aligned} 2\pi \| \; u_n - u(n\Delta t) \; \|^2_{L^2} &= \| \; (\hat{G}^n - \hat{H}^n)\hat{u}_0 \; \|^2_{L^2} \\ &= \| \; (\hat{G}^{n-1} + \hat{G}^{n-2}\hat{H} + \ldots + \hat{H}^{n-1})(\hat{G}-\hat{H})\hat{u}_0 \; \|^2_{L^2} \\ &\leq \| \; \hat{G}^{n-1} + \hat{G}^{n-2}\hat{H} + \ldots + \hat{H}^{n-1}\|^2_{L^\infty} \|(\hat{G}-\hat{H})\hat{u}_0 \; \|^2_{L^2} \\ &\leq (Kn(\Delta t)^2)^2 \leq (KT\Delta t)^2, \end{aligned}$$

and consequently the convergence rate is $\mathcal{O}(\Delta t)$. □

Let us study the relations between the operators G and H for the simple model problem

$$\begin{aligned} u' + \lambda u &= 0 \\ u(0) &= 1 \end{aligned}$$

with an approximate solution $u_{n+1} = r(x)u_n$ (where $x = \lambda\Delta t$):

(1) the exact solution satisfies

$$r(x) = e^{-\lambda\Delta t} = e^{-x},$$

(2) the forward Euler method

$$\frac{u_{n+1} - u_n}{\Delta t} + \lambda u_n = 0 \quad \Rightarrow \quad r(x) = 1 - x,$$

(3) the backward Euler method

$$\frac{u_{n+1} - u_n}{\Delta t} + \lambda u_{n+1} = 0 \quad \Rightarrow \quad r(x) = (1 + x)^{-1},$$

(4) the trapezoidal method

$$\frac{u_{n+1} - u_n}{\Delta t} + \frac{\lambda}{2}(u_n + u_{n+1}) = 0 \quad \Rightarrow \quad r(x) = \left(1 + \frac{x}{2}\right)^{-1} \left(1 - \frac{x}{2}\right),$$

and

(5) the Lax-Wendroff method

$$u_{n+1} = u_n - \Delta t \lambda u_n + \frac{1}{2}\Delta t^2 \lambda^2 u_n \quad \Rightarrow \quad r(x) = 1 - x + \frac{1}{2}x^2.$$

The consistence $|e^{-\lambda\Delta t} - r(\lambda\Delta t)| = \mathcal{O}(\Delta t^{p+1})$ holds with $p = 1$ in case 2 and 3, and $p = 2$ in case 4 and 5. The following stability relations hold:

(1) $|r(x)| \leq 1$ for $x \geq 0$ in case 1, 3 and 4.

(2) $r(x) \to 0$ as $x \to \infty$ in case 1 and 3.

(3) $r(x) \to 1$ as $x \to \infty$ in case 4.

Property (1) shows that for $\lambda > 0$ case 3 and 4 are unconditionally stable. However Property (2) and (3) refine this statement and imply that only case 3 has the same damping behavior for large λ as the exact solution. Although the damping Property (2) is not necessary to prove convergence it is advantegous to have for proplems with many time scales, e.g. for a system of equations (6.3) where A has eigenvalues $\lambda_i \leq 1, \; i = 1, \ldots, N$ and some $\lambda_j \ll -1$, (why?).

The unconditionally stable methods, e.g. case 3 and 4, are in general more efficient to solve parabolic problems, such as the Black-Scholes equation (6.2), since they require for the same accuracy fewer time steps than the explicit methods, e.g. case 2 and 5. Although the work in each time step for the unconditionally stable methods may be larger than for the explicit methods.

Exercise 6.7 Show by an example that $\|u_n\|_{L^2}^2 \to \infty$ if for some ω there holds $|\hat{G}(\omega)| > 1$, in Example 6.6, i.e. the von Neumann stability condition does not hold.