1 Hamilton–Jacobi equations

A Hamilton–Jacobi differential equation (one space dimension) is of the form
\[ q_t + f(q_x) = 0 \]
It is related to a conservation law by \( v = q_x \),
\[ v_t + f(v)_x = 0 \]
so methods for conservation laws are useful. But the conservation law is obtained by differentiation, so needs other initial (and boundary) conditions.

Conservation laws are mathematical expressions of basic facts like mass and energy conservation, whereas Hamilton–Jacobi equations are related to optimization and optimal control problems. Lab 6 treats the problem of finding the fastest way from every point in a domain to its boundary.

2 Short wave asymptotics: The Eikonal equation and ray-tracing

We will develop here the eikonal equation which is a common short-wavelength asymptotic approximation to wave propagation problems. Consider a time-harmonic solution
\[ q = e^{i\omega t}v(x, y) \]
to the wave equation
\[ q_{tt} = c^2(x, y)(q_{xx} + q_{yy}) \]
giving the Helmholtz equation
\[ k^2(x, y)v + \Delta y = 0, \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]
in a domain of size \( L \). The wave-number is defined as
\[ k = \frac{\omega}{c(x, y)} = \frac{2\pi}{\lambda(x, y)} \]
The wave (phase) speed \( c \) and the wavelength \( \lambda \) depend on \((x, y)\). For accuracy, the discretization must resolve the wavelength, with \( O(10) \) cells per wavelength. A typical example would be calculation of reflections from a 10 m object of 10 mm microwaves, requiring about \( 10^4 \) cells in each of three space dimensions, or \( 10^{12} \) cells. This is infeasible on today’s hardware.

Now, let us represent the solution by its phase \( k_0S(x, y) \) (the reference wave number \( k_0 \) is defined below) and amplitude \( A(x, y) \), both real,
\[ v = A(x, y)e^{ik_0S(x, y)} \] (*)
and one derives (Exercise: do it!) the equations
\[ 2\nabla A \cdot \nabla S + A\Delta S = 0 \]
\[ -k_0^2|\nabla S|^2A + k^2A + \Delta A = 0 \]
as imaginary and real parts of the Helmholtz equation with (*). The first line is the “transport equation” for the amplitude, which may be written
\[ \nabla \cdot (A^2 \nabla S) = 0 \]
from which one can compute \( A \) once \( S \) is known.

When \( k \) (and \( k_0 \)) are large, the two first terms of the second line must balance, almost everywhere, and the result is the Eikonal equation
\[ |\nabla S|^2 = n^2(x, y), n = \frac{k(x, y)}{k_0} = \frac{c_0}{c(x, y)} \]
where \( n \) is called the index of refraction, and \( c_0 \) is the wave speed in a reference medium, like vacuum (for electromagnetic waves). For electromagnetic waves in the visual spectrum, in most materials, \( n \) is between 1 and 3, but “esoteric” active materials may be designed with a larger range of \( n \).
3 Eikonal equation and shortest distance

To understand the solutions, we first consider $n = 1$, a constant. Then,

$$S(p) - S(q) = \int_0^1 \nabla S(p + t(q - p)) \cdot (p - q) dt \cdot \cos \theta(t) dt \leq |p - q|$$

On the other hand, we may trace a ray from $p$ by following the gradient of $S$:

$$dX/ds = \nabla S(X), X(0) = p$$

Since the modulus of $dX/ds$ is one, $s$ must be the arclength along the ray. Now, for an $X$ on the ray,

$$S(X(s)) - S(p) = \int_0^s \nabla S(X(t)) \cdot dX/dt dt \geq |X(s) - p|$$

and the two inequalities show that

$$S(X(s)) - S(p) = |X(s) - p|$$

so the rays are straight lines. Let us set $S = 0$ on a closed curve $\Gamma$, and trace rays from all points on $\Gamma$, following the normals to the curve, giving values of $S$ along the rays. As long as the normals do not cross, all points on an iso-curve $L$ of $S$ have the same distance to $\Gamma$, and that is the shortest distance from $L$ to $\Gamma$. If the normals cross, the ray construction leads to a multi-valued $S$; we define a unique $S$ as the smallest of the different possible values. It is then clear that:

The solution $S(x,y)$ to the Eikonal equation, with boundary value $S = 0$ on $\Gamma$, is the shortest distance from $(x,y)$ to $\Gamma$.

3.1 Variable $n$

If the index of refraction varies, the rays are no longer straight, and the solution instead gives the shortest times for traveling from $(x,y)$ to $\Gamma$. It is still possible to trace the rays and construct $S$. We have, for each coordinate $i$ of $X$,

$$\frac{dX_i}{dt} = \frac{\partial S}{\partial x_i}, \quad \frac{d^2 X_i}{dt^2} = \sum_{j=1}^N \frac{\partial^2 S}{\partial x_j \partial x_i} \frac{dX_j}{dt} = \sum_{j=1}^N \frac{\partial S}{\partial x_j} \frac{\partial^2 S}{\partial x_i \partial x_j} = \frac{1}{2} \frac{\partial}{\partial x_i} \sum_{j=1}^N \left( \frac{\partial S}{\partial x_j} \right)^2$$

or, the ray-tracing equation,

$$\frac{d^2 X}{dt^2} = n \nabla n$$

Taking the scalar product of the ray-tracing equation with $dX/dt$ gives

$$\frac{d^2 X}{dt^2} \cdot \frac{dX}{dt} = \frac{1}{2} \nabla n \cdot \frac{dX}{dt} = \frac{1}{2} \frac{dn^2}{dt} (X(t))$$

$$\frac{1}{2} \frac{d}{dt} \left( \left| \frac{dX}{dt} \right|^2 - n^2 \right) = 0 \Rightarrow \left| \frac{dX}{dt} \right| = n$$

so $dt = ds/n$ where $ds$ is the arclength element along the ray. This explains the shortest travel time property.

3.2 Snell’s law of refraction

Consider the common case of piecewise constant $n$. In a region of constant $n$, the rays are straight, and at discontinuities of $n$, they turn with a discontinuous tangent, following Snell’s law of refraction,

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

where $\theta$ is the angle made by the ray with the normal plane of discontinuity; the incoming and outgoing rays and the normal are in a plane, the $(x,y)$-plane with $y$ the normal direction.

Proof:

Since $n$ varies only with $y$, we have
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\[ \frac{d^2 x}{dt^2} = 0 \Rightarrow \frac{dx}{dt} = u = \text{const.}, \frac{dy}{dt} = \frac{u}{\tan \theta} \]

\[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = u^2 + \left( \frac{u}{\tan \theta} \right)^2 = n^2 = \frac{u^2}{\sin^2 \theta} \]

Example: Sound in stratified media.
Suppose that \( n(y) \) has a sinusoidal variation,
\( n = 1 + a \cos y \)
Rays are emitted at the origin. Calculate the ray shapes.
\[ \frac{d^2 x}{d\tau^2} = 0, \frac{d^2 y}{d\tau^2} = -a \sin y \]
We get \( x = b\tau \) (some \( b \), depending on the launch angle \( \alpha \)) and
\[ \frac{d^2 y}{dx^2} = -\frac{a}{b^2} \sin y \]
For small \( \alpha \), we obtain
\[ y(x) = \tan \alpha \cdot \frac{b}{\sqrt{a}} \sin \frac{x\sqrt{a}}{b} \]
Note that all these rays cross at
\[ y = 0, x_m = \frac{mb\pi}{\sqrt{a}}, m = 1, 2, ... \]
the convergence zones, well known from submarine movies.

4 The H-J Eikonal equation
The Eikonal equation is non-linear and it is not immediately clear how to discretize and solve it numerically. Introducing a “time” \( t \) (with dimension length, so is actually distance), consider
\[ S_t + \nabla S - n(x, y) = 0 \]
which we will use to solve the Eikonal equation numerically by explicit schemes for conservation laws.

4.1 Analysis of linearization
A perturbation \( s \) to the steady solution \( S \) (the existence of which we assume for now) satisfies (The verification is left for the Homework)
\[ s_t + U \cdot \nabla s = 0, \quad U = \frac{\nabla S}{|\nabla S|} \]
which is hyperbolic, and the boundary/initial value problem with boundary condition \( s = 0 \) on \( \Gamma \) is well-posed. Given some initial condition, the steady state (\( s = 0 \)) at a point \( P \) is reached in finite “time” equal to the shortest distance between \( P \) and \( \Gamma \), because the velocity \( U \) has magnitude 1. An explicit time stepping scheme can compute the solution in time proportional to the number of gridpoints \( x \) max number of cells from a point to \( \Gamma \).
The Lax-Friedrichs scheme is
\[ S_{ij}^{n+1} - \frac{1}{4} (S_{i+1,j}^n + S_{i-1,j}^n + S_{i,j+1}^n + S_{i,j-1}^n) = \frac{\Delta t}{2\Delta x} \left[ \sqrt{(S_{i+1,j}^n - S_{i,j}^n)^2 + (S_{i,j+1}^n - S_{i,j}^n)^2} - n(x_i, y_j) \right] \]
If used on the linearized problem, with velocity \( U = (\cos \alpha, \sin \alpha) \) it has von Neumann amplification factor
\[ G = \frac{1}{2} (\cos \theta_x + \cos \theta_y) - i \frac{\Delta t}{\Delta x} (\sin \theta_x, \sin \theta_y) \cdot (\cos \alpha, \sin \alpha), \quad |\theta_x|, |\theta_y| \leq \pi, \]

… to be finished in the homework.