Examination paper Numerical Solution of DE, 2D1255

Suggested answers

P1. (a) Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$. Its eigenvalues are $\lambda_{1,2} = \frac{1}{2}\left(a \pm \left(a^2 + bc\right)^{1/2}\right)$, real, hence a hyperbolic system, if $a^2+bc \geq 0$. Symmetric if $b = c$.

(b)

$\mathbf{Q}_j^n = \mathbf{P}^n(k)e^{ikj\Delta x}$ gives, with $\theta = k\Delta x$

$$\mathbf{P}^{n+1} = \left[\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) - \frac{\Delta t}{2\Delta x}\mathbf{A}\left(e^{i\theta} - e^{-i\theta}\right)\right]\mathbf{P}^n \text{ so}$$

$\mathbf{G} = \mathbf{I}\cos\theta - i\frac{\Delta t}{\Delta x}\mathbf{A}\sin\theta$ with eigenvalues $\mu_{1,2} = \cos\theta - i\frac{\Delta t}{\Delta x}\lambda_{1,2}\sin\theta$

We have $g_{\max} = \max\limits_{0\leq\theta<\pi}\left|\mu_{1,2}\right| = \max(1, \max\limits_{k=1,2}\left|\frac{\Delta t}{\Delta x}\lambda_k\right|)$

G has the same eigenvectors as **A**, independent of θ. It is therefore enough to require $g_{\max} \leq \left(1 + K\Delta t\right), K$ independent of $\Delta t, \Delta x$. Since the ratio $\Delta t/\Delta x$ must be bounded above and below as we send Δx to 0, that is only possible if

$$g_{\max} = \max_{k=1,2}\left|\frac{\Delta t}{\Delta x}\lambda_k\right| \leq 1$$

(c) i) The *CFL "observation"* is that convergence is impossible if the numerical domain of dependence does NOT include the mathematical domain of dependence. The CFL condition is the requirement on $\Delta t/\Delta x$ for the numerical domain of dependence to INCLUDE the mathematical domain of dependence.

ii) the *Lax Equivalence Theorem* states that convergence and stability (in the Lax sense) are equivalent for consistent schemes.

iii) For consistent schemes, therefore, the CFL condition is a *necessary* condition for *stability*.

P2. (a) See Leveque, (b) See Leveque

P3. (a)

H, U constant satisfy the full equation. With $h = H + \delta h, u = U + \delta u$,

$$\delta h_t + (U + \delta u)(\underbrace{H_x}_{0} + \delta h_x) + (\underbrace{U_x}_{0} + \delta u_x)(H + \delta h) = \underline{\delta h_t + U\delta h_x + H\delta u_x} + O(\delta^2) = 0$$

and ... $\delta u_t + U\delta u_x + g\delta h_x = 0$

which we write $\begin{pmatrix} \delta h \\ \delta u \end{pmatrix}_t + \mathbf{A}\begin{pmatrix} \delta h \\ \delta u \end{pmatrix}_x = 0, \mathbf{A} = \begin{pmatrix} U & H \\ g & U \end{pmatrix}$ with eigenvalues $\lambda_{\pm} = U \pm c, c = \sqrt{gH}$

(b) We count the characteristics that run *into* the computational domain at each boundary, which is also the *number* of (scalar) conditions that must be given. Also the condition must specify the values of the characteristic variables associated with the ingoing characteristics, which must be considered in case 2 below.

Three cases:

	$x = 0$	$x = 1$
1. $U\text{-}c < U\text{+}c < 0$	0	2
2. $U\text{-}c < 0 < U\text{+}c$	1	1
3. $0 < U\text{-}c < U\text{+}c$	2	0

P4. (8)
(a and e)
Let $\mathbf{A}$ be diagonalizable, $\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1}$ where $\mathbf{S}$ is the matrix of right eigenvectors $\mathbf{r}_k$
$\mathbf{A}\mathbf{r}_k = \lambda_k \mathbf{r}_k, \Lambda = diag(\lambda_k)$. Then
$\left|\mathbf{A}\right| = \mathbf{S}diag(\left|\lambda_k\right|)\mathbf{S}^{-1}, \mathbf{A}^+ = 1/2\left(\mathbf{A} + \left|\mathbf{A}\right|\right), \mathbf{A}^- = 1/2\left(\mathbf{A} - \left|\mathbf{A}\right|\right)$
If all eigenvalues have $\left|\lambda_k\right| = c, \left|\mathbf{A}\right| = \mathbf{S}c\mathbf{I}\mathbf{S}^{-1} = c\mathbf{I}$
(b) A time-stepping scheme in conservation form is

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n)$$

(c) Suppose the numerical flux $\mathbf{F}_{j-1/2}^n$ is an $\alpha+\beta+1$-point scheme calculated from the $\mathbf{Q}_k$, $k = j\text{-}\alpha, \ldots, j+\beta$, $\mathbf{F}_{j-1/2} = \mathbf{Z}(\mathbf{Q}_{j-\alpha},\ldots,\mathbf{Q}_{j-1},\mathbf{Q}_j,\ldots,\mathbf{Q}_{j+\beta})$. Then, in order that the scheme approximate the correct differential equation, the function $\mathbf{Z}$ must satisfy

$$\mathbf{Z}(\mathbf{Q},\ldots,\mathbf{Q},\mathbf{Q},\ldots,\mathbf{Q}) = \mathbf{f}(\mathbf{Q})$$

as well as Lipschitz-conditions in all arguments.
(d) The Roe scheme has numerical flux

$$\mathbf{F}_{j-1/2}^n = \frac{1}{2}\left(\mathbf{f}(\mathbf{Q}_{j-1}^n) + \mathbf{f}(\mathbf{Q}_j^n)\right) - \frac{1}{2}\left|\mathbf{A}_{j-1/2}^n\right|\left(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n\right)$$

The “flux Jacobian matrix” is the $s \times s$ matrix of partial derivatives $\mathbf{J}(\mathbf{q}) = \left(\frac{\partial f_i}{\partial q_j}\right)$

$\mathbf{A}$ is the Roe-matrix, which satisfies $\mathbf{A}_{j-1/2}\left(\mathbf{Q}_j - \mathbf{Q}_{j-1}\right) = \mathbf{f}(\mathbf{Q}_j) - \mathbf{f}(\mathbf{Q}_{j-1})$. For the shallow water eq’ns and the Euler eqn’s of gas dynamics $\mathbf{A}$ can be computed as $\mathbf{J}$ evaluated at a specific average (the Roe average) of $\mathbf{Q}_j$ and $\mathbf{Q}_{j-1}$.

P5. (8)
(a) *prolongation* means extending a grid-function on a coarse grid to the fine grid, by some interpolation scheme; and *restriction* means the reverse, to set values on the coarse grid by suitable sub-sampling or other interpolation procedure.
(b) Let us define finite difference grids so that the boundary gridpoints match exactly on all grids, and the stepsize is halved between successive grid levels. Let us further assume Dirichlet conditions and NOT count the boundary points among the unknowns. Thus, the linear interpolation prolongation is a 7 by 3 matrix:

$$\mathbf{P}=\begin{pmatrix} .5 & 0 & 0 \\ 1 & 0 & 0 \\ .5 & .5 & 0 \\ 0 & 1 & 0 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \\ 0 & 0 & .5 \end{pmatrix}, \mathbf{R}_{inject}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{R}_{full}=\frac{1}{2}\begin{pmatrix} .5 & 1 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & 1 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 1 & .5 \end{pmatrix}=\frac{1}{2}\mathbf{P}^T$$

(c) The damped Jacobi iteration when written for $w_j^n = U_j - u_j^n$ i.e. $u_j^n = U_j - w_j^n$ becomes

$$U_j - w_j^{n+1} = (1-\alpha)(U_j - w_j^n) + \alpha\left[-\Delta x^2 f_j + (U_{j-1} - w_{j-1}^n) + (U_{j+1} - w_{j+1}^n)\right]/2$$

$$\underbrace{U_j - (1-\alpha)U_j - \alpha\cdot\left[-\Delta x^2 f_j + U_{j-1} + U_{j+1}\right]/2}_{=0} - w_j^{n+1} = -(1-\alpha)w_j^n + \alpha\left[-w_{j-1}^n - w_{j+1}^n\right]/2$$

$$w_j^{n+1} = (1-\alpha)w_j^n + \alpha\left[w_{j-1}^n + w_{j+1}^n\right]/2$$

$$w_j^0 = U_j,$$

where the terms indicated cancel because U_j is the exact solution to the discretized equation.is applied. The von Neumann ansatz

$w_j^n = r^n \cdot e^{ikj\Delta x} = r^n \cdot e^{ij\theta}, \theta = k\Delta x, 0 < \theta < \pi$ gives

$r^{n+1} = (1-\alpha)r^n + \frac{\alpha}{2}\left(e^{i\theta} + e^{-i\theta}\right)r^n$, so $G = (1-\alpha) + \alpha\cos\theta = 1 - 2\alpha\sin^2\frac{\theta}{2}$

It follows that $0 < \alpha < 1$ is necessary for $|G|$ to be smaller than 1.