Examination paper Numerical Solution of DE, 2D1255/DN2255

14-19, May 20, 2008

Closed books examination. Read all the questions before starting work. Check carefully that your initially derived equations are correct. Ask if you are uncertain. Answers MUST be motivated. Paginate and write your name on EVERY page handed in.
A total of 21 out of max 44 points guarantees a "pass". The results will be e-mailed to participants by June 1, 2008. Papers are kept at the CSC Student Office for a year and then destroyed. Complaints to J.Oppelstrup by Sep. 1, 2008, after which the results are irrevocable. The next examination paper will be given in September, 2008.

P1. (8)

(4) (a) The 1-D elliptic problem $u_{xx} = f(x), u(0) = u(1) = 0$ is discretized by central differences into $u_{j-1} - 2u_j + u_{j+1} = \Delta x^2 f(x_j), j = 1,\ldots,n, u_0 = u_{n+1} = 0$ with solution U_j. The "time stepping"

$$u_j^{n+1} = u_j^n + \alpha(-\Delta x^2 f_j + u_{j-1}^n + u_{j+1}^n - 2u_j^n)$$
$$u_j^0 = 0,$$

is applied as smoother. Write the iteration for the error $w_j^n = U_j - u_j^n$. Use von Neumann analysis with ansatz $w_j^n = \hat{w}^n \cdot e^{ikj\Delta x} = \hat{w}^n \cdot e^{ij\theta}, \theta = k\Delta x$, and find the amplification factor G. What range of θ must we consider? What are the "high"- and "low"-frequency ranges?

(4) (b) Plot the amplification factor $G(\alpha,\theta)$ vs. θ (for some α) and find the choice of α, which will give

$$\text{i) } \min_\alpha\left(\max_{0<\theta<\pi} |G(\alpha,\theta)|\right) \quad \text{ii) } \min_\alpha\left(\max_{\pi/2<\theta<\pi} |G(\alpha,\theta)|\right)$$

Which, if any, of these is the choice for the multi-grid method? Explain!

P2. (9)

Consider the initial-boundary value problem on the strip $0 \le x \le 1$, $t \ge 0$:

$$\mathbf{q}_t + \begin{pmatrix} 1 & U \\ U & 1 \end{pmatrix} \mathbf{q}_x = 0, \mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}$$
$$v = 0 \text{ at } x = 0$$
$$u + \alpha v = 0 \text{ at } x = 1$$
$$u(x,0) = \exp(-500(x-0.5)^2), v(x,0) = 0$$

(α real; A short pulse centered at $x = 0.5$, details not important)

(3) a) Diagonalize the system and find the characteristic speeds and variables (usually called w1 and w2.)

(2) b) For which values of U can one boundary condition at $x = 0$ and one at $x = 1$ give a well-posed problem?

(4) c) For the range of U found in b), choose α (may depend on U) so there are no reflections at $x = 1$.
Hint: Eigenvectors!

P3. (6)

(3) (a) Derive a well-known method for the advection equation $q_t + uq_x = 0$ from the following algorithm.

1) Reconstruct a piecewise constant function $\tilde{q}^n(x)$ from cell averages Q_j^n

2) Solve the Riemann problem and find the values of q, $q^*_{i+1/2}$, at cell interfaces $x_{i+1/2}$ at t^n+

3) Compute the cell averages at t^{n+1} by

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right), \quad F_{j+1/2}^n = uq_{j+1/2}^*$$

What is the scheme?

(3) (b) Discuss how the reconstruction step can be modified to obtain a high resolution method. You need to explain the use of "slope limiters".

P4. (6)

(3) (a) Linearize the isothermal gas dynamics equations

$$\begin{cases} \rho_t + u\rho_x + u_x\rho = 0 \\ \rho(u_t + uu_x) + a^2\rho_x = 0 \end{cases}$$

where ρ = density, u = velocity, and constant $a^2 > 0$, around constants $\rho = R$, $u = U$. What are the characteristic speeds? What is the meaning of a?

(3) (b) Consider an initial-boundary value problem on $0 < x < 2$. Let $a = 10$ and $U = 4$, and the perturbation at $t = 0$ be

$$\delta\rho(x,0) = 0.01R \cdot \exp(-500(x-0.5)^2), \delta u(x,0) = 0$$

At what times do the pulse(s) first hit the boundary i) $x = 0$ ii) $x = 2$?
When the Lax-Friedrichs scheme is used on a grid with 100 cells, what is the largest stable time-step? *Hint:* max. characteristic speed.

P5. (7)

(2) (a) Define the meaning of $\mathbf{A}^+$, $\mathbf{A}^-$, and $|\mathbf{A}|$ as used in the Roe-scheme. $\mathbf{A}$ is a real square matrix.
Let the system of conservation laws be $\mathbf{q}_t + (\mathbf{f}(\mathbf{q}))_x = 0$.
A time-stepping scheme in conservation form is

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{j+1/2}^n - \mathbf{F}_{j-1/2}^n)$$

(2) (b) The numerical flux $\mathbf{F}_{j-1/2}^n$ should satisfy a consistency condition … which?

(1) (c) The Roe scheme has numerical flux

$$\mathbf{F}_{j-1/2}^n = \frac{1}{2}\left(\mathbf{f}(\mathbf{Q}_{j-1}^n) + \mathbf{f}(\mathbf{Q}_j^n)\right) - \frac{1}{2}\left|\mathbf{A}_{j-1/2}^n\right|\left(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n\right)$$

Define the "flux Jacobian matrix" and its relation to the Roe matrix $\mathbf{A}_{j-1/2}$.

(2) (d) Suppose all eigenvalues of $\mathbf{A}$ have equal magnitude, c. Then $|\mathbf{A}|$ becomes very simple. What?

P6. (8)

(2) Explain the CFL non-convergence criterion for a three-point consistent scheme for a hyperbolic system,

(2) Explain the Lax equivalence which relates consistency, (Lax-Richtmyer) stability, and convergence. Also define the concepts.

(4) What are the Rankine-Hugoniot (aka. shock speed) relations for

a) the Burgers’ equation

$$q_t + q q_x = 0$$

b) isothermal gas dynamics

$$\begin{cases} \rho_t + u\rho_x + u_x\rho = 0 \\ \rho(u_t + uu_x) + K\rho_x = 0 \end{cases}$$

Hint: You must first write the equations in conservation form; for gas dynamics the conserved variables are ρ and $m = \rho u$. Hint: Add the mass conservation equation multiplied by u to the momentum equation.