Examination paper Numerical Solution of DE, 2D1255/DN2255

8-13, May 23, 2011

Closed books examination. Read all the questions before starting work. Check carefully that your initially derived equations are correct. ***Ask if you are uncertain*** !.
Answers MUST be motivated. You can judge the level of details required in the answers from the number of points. Paginate and write your name on EVERY page handed in.
A total of $N/2$ out of max N (= 50) points guarantees a "pass". The results will be e-mailed to participants by June 2, 2011. Papers are kept at the CSC Student Office for a year and then destroyed. Complaints to J.Oppelstrup by June 23, 2011, after which the results are irrevocable.

P1 (9) – Parabolic problems

Consider the orthotropic initial-boundary value heat conduction problem for $t > 0$ on the unit square,

$$q_t = a(y)q_{xx} + b(x)q_{yy}, \quad a(.), b(.) \ge 0$$
$$q(x,y,0) = f(x,y)$$
$$q_x(0,y,t) = q_x(1,y,t) = 0$$
$$q_y(x,0,t) = q_y(x,1,t) = 0$$

a) (3) Show that the total amount of "heat" is conserved,

$$\int_\Omega q(x,y,t)dxdy = \int_\Omega f(x,y)dxdy$$

b) (3) Show that the L2-norm of $q(.,.,t)$ is non-increasing,

$$\int_\Omega q^2(x,y,t)dxdy \le \int_\Omega f^2(x,y)dxdy$$

Hint: Integration by parts.

c) (3) The tridiagonal nxn matrix **T** with diagonal -2 and elements $T_{i,i+1}$ and $T_{i,i-1} = 1$ plays an important role when the boundary conditions are $q = 0$. Its m:th eigenvector $\mathbf{r}^m$, $m = 1,2,\ldots,n$, has elements $r_k^m = \sin\frac{km\pi}{n+1} = \mathrm{Im}\left(\exp(i\frac{km\pi}{n+1})\right), k = 1,2,...,n$

Don't prove that but *do* compute the eigenvalues λ_m !

P2 (9) - Linear and nonlinear hyperbolic systems

Consider the Riemann problem for a discontinuity with left state q_L and right state q_R at (t=0,x=0) for the conservation law

$$q_t + f(q)_x = 0, f(q) = q^2$$

(a) (3) Determine for which part of the (q_L,q_R)-plane the solution is a *shock*. Hint: Lax' condition of shock collision. What kind of solution for the rest? You need not prove your answer, which follows from the fact that $f(.)$ is convex.

(b) (3) For the shock solution, determine the shock speed and the "Godunov state" $q^{\vee\!|}$ (Leveque's notation), i.e. $q(x=0,t=0+)$.

(c) (3) For the "other kind of solution", find $q(x,t)$ as a similarity solution

$$q(x,t) = v(\xi), \xi = x/t$$

and from that find $q^{\vee\!|}$ for these (q_L,q_R)

P3 (10) - Well-posedness, stability and other properties of numerical schemes

Consider a Cauchy-problem for the first order system (a,b,c real)

$$\mathbf{q}_t + \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\mathbf{q}_x = 0 \qquad \text{(S)}$$

a) (2) What is the condition on a,b,c that the system be hyperbolic? symmetric hyperbolic?
The Lax-Friedrichs scheme for a constant coefficient hyperbolic first order system $\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0$ is

$$Q_j^{n+1} = \frac{1}{2}(Q_{j+1}^n + Q_{j-1}^n) + \frac{\Delta t}{2\Delta x} A(Q_{j+1}^n - Q_{j-1}^n)$$

b) (3) Compute the von Neumann magnification matrix **G** for an ansatz

$$Q_j^n = U^n e^{ijk\Delta x}$$

c) (3) For $a = 8$, $b = 1$, $c = 9$, what are the eigenvalues of G? For which values of Δt and Δx is the method stable?

d) (2) Now consider an initial-boundary value problem on $0 < x < L$ for (S). Show that, for $bc > 0$, boundary conditions are necessary at $x = 0$ *and* $x = L$ for well-posedness.

P4 (9) – The Roe Scheme

Consider the system of conservation laws $\mathbf{q}_t + (\mathbf{f}(\mathbf{q}))_x = 0$. A time-stepping scheme in conservation form is

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1/2}^n - \mathbf{F}_{j-1/2}^n\right)$$

The Roe scheme has numerical flux

$$\mathbf{F}_{j-1/2}^n = \frac{1}{2}\left(\mathbf{f}(\mathbf{Q}_{j-1}^n) + \mathbf{f}(\mathbf{Q}_j^n)\right) - \frac{1}{2}\left|\mathbf{A}_{j-1/2}^n\right|\left(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n\right)$$

a) (2) Define the meaning of $\mathbf{A}^+$, $\mathbf{A}^-$, and $|\mathbf{A}|$ as used in the Roe-scheme. **A** is a real square matrix.

b) (3) The Roe matrix satisfies

$$\mathbf{A}_{j-1/2}^n\left(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n\right) = \mathbf{f}(\mathbf{Q}_j^n) + \mathbf{f}(\mathbf{Q}_{j-1}^n) \qquad (*)$$

Show that the update formula can be written (using (*))

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left(\mathbf{A}_{j-1/2}^+(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n) + \mathbf{A}_{j+1/2}^-(\mathbf{Q}_{j+1}^n - \mathbf{Q}_j^n)\right)$$

c) (4) Show that the Roe matrix (… 1 x 1) exists for scalar equations of the form $q_t + (P(q))_x = 0$ where P is a polynomial of degree N, $P(q) = \sum_{k=0}^{N} c_k q^k$ by giving a formula. First show that

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$$

P5 (13) - Spectral methods

Consider the spectral differentiation method applied to the $2A$ periodic initial value problem on $[-A,A]$

$$q_t + aq_x = cq_{xx}, c > 0$$

with simple explicit Euler discretization in time, and an even number N of gridpoints,

$$x_k = k\Delta x, k = -N/2, -N/2+1, \ldots, 0, 1, 2, \ldots, N/2-1, \Delta x = 2A/N.$$

Here are formulas for the discrete, symmetric Fourier transform and its inverse:

$$w_m = \frac{1}{N}\sum_{\alpha=-N/2}^{N/2-1} q_\alpha e^{-i\pi m x_\alpha / A}, q_\alpha = \sum_{m=-N/2}^{N/2-1} w_m e^{i\pi m x_\alpha / A}$$

a) (4) Show that the time evolution of the Fourier coefficients becomes

$$w_m^{n+1} - w_m^n = \Delta t(-ai\mu w_m^n - c\mu^2 w_m^n), \frac{\mu A}{\pi} = m = -N/2, \ldots, N/2-1 \ (*)$$

In the following, you may use (*) also if you did not derive it.

b) (2) What is the von Neumann growth factor G?

c) (3) Show that von Neumann stability requires the time-step limit

$$\Delta t \le \frac{2c}{a^2 + c^2 (\frac{\pi N}{2A})^2}$$

For $a = 0$, compare to the central difference stability limit $\frac{c\Delta t}{\Delta x^2} \le \frac{1}{2}$

d) (4) The transform $\{w_m\}$ of a numerical “delta-function”, $q_\alpha = \begin{cases} 1, \alpha = k \\ 0, \alpha \ne k \end{cases}$

is $w_m = \frac{1}{N} e^{-i2\pi m \frac{k}{N}}$. Show that $I(x) = \sum_{m=-N/2}^{N/2-1} w_m e^{i\pi m x / A}$

satisfies $I(x_\alpha) = \begin{cases} 1, \alpha = k \\ 0, \alpha \ne k \end{cases}$, so I is the Fourier interpolant.

Hint You can use the formula for a finite geometric series without proof,

$$\sum_{m=-N/2}^{N/2-1} e^{imy} = e^{-i\frac{y}{2}} \frac{\sin(Ny/2)}{\sin(y/2)}$$