Homework 3: Well-posedness, stability and other properties of numerical schemes Max. 4 p.

Topics
Well-posedness and stability by Fourier analysis; Lax-Richtmyer stability; Modified equations, dissipation and dispersion.
Purpose
To get acquainted with analysis techniques for solution schemes for initial-boundary value problems for hyperbolic systems
Instructions
Write a short report with the plots and answers to the questions posed. Make sure the plots are annotated and there is explanation for what they illustrate.

1 Well-posedness and von Neumann analysis

Consider 2π-periodic Cauchy problems in the cases $\alpha = 1$ and $\alpha = -1$ for

$$u_t = \alpha u_{xxxx},$$
$$u(x,0) = \sin x.$$

a) (0.5p) Investigate the well-posedness in L_2 of both cases by theoretical considerations. You may assume the existence of a solution and just do an energy estimate.

Note: To show ***non***-well-posed-ness you must *find a family of solutions with unity L_2 norm of the initial data whose growth rate is unbounded*.

b) (0.5p) Use a finite difference approximation with central differences in space and forward difference in time. Apply von Neumann analysis to determine how Δt and Δx should be related for the method to be stable in the well-posed case(s). Is there a stable discretization of the ill-posed case(s)?

c) (0.5p) Implement the scheme in a Matlab code and illustrate the conclusions by numerical experimentation.

2 L_1 Lax-Richtmyer stability and the modified equation

Consider the Lax-Friedrichs method applied to the Cauchy problem ($-\infty < x < +\infty$) for the advection equation,

$$u_t + a u_x = 0$$

with initial L_1 - data $u(x,0)$.

a) (0.5p) Show that the method is Lax-Richtmyer stable in the L_1-norm if the CFL condition ("numerical domain of dependence includes mathematical domain of dependence") is satisfied.

b) (0.5p) Derive the modified equation and discuss how to choose Δt and Δx to *avoid* strong damping effects. (You may do this for the finite difference version, rather than the finite volume version.)

3 Shallow water equations, dissipation and dispersion

a) (0.5p) Run your Lax-Friedrichs program for Homework 2 with different values of $\Delta t/\Delta x$. How large can Δt be without violating the CFL condition?
Plot solutions for different $\Delta t/\Delta x$. When is the damping of waves largest/smallest? Compare with your prediction in the previous question.

b) (1.0p) Write a similar program that solves the problem by the two-step Lax-Wendroff method known as the McCormack scheme on page 337 in LeVeque. According to section 8.6.2 the method is dispersive.

1) Hand in plots of the solution where effects of dispersion errors can be seen. *Don't use the "magic timestep"*.

2) How is this effect changed by higher spatial resolution? Do you see any damping?

3) Compare (disspation/dispersion) with the solution obtained with the Lax-Friedrichs method.