# Boundary integral equation methods for elliptic problems

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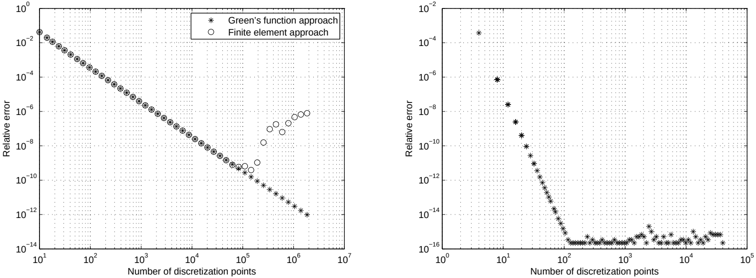
# Overview

- Why integral equation methods and when do they apply?
- Integral equation formulations of PDEs.
- Numerical solution
- Fast numerical solution
- Some shortcomings
- Wrapping up

# Why integral equation methods?

- The basic idea is to rewrite a PDE as an integral equation.
- So, we go from differentiation to integration.
- High accuracy integration is a lot easier than high accuracy differentiation.
- Condition numbers stay bounded with integral equations.
- Example problem :

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \sin(\pi x), \qquad x \in [-1, 1], \qquad u(-1) = u(1) = 0.$$



# Why integral equation methods?

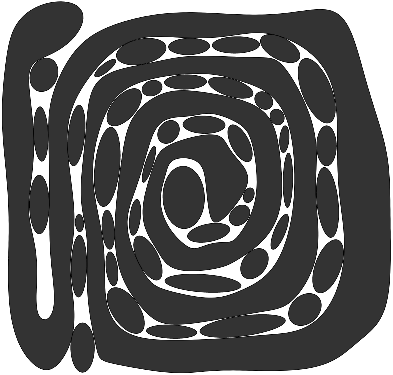
- In some cases the solution of the PDE in the interior of a domain $\Omega$ is completely determined by the boundary conditions on $\partial\Omega$.
- An example is Laplace's equation :

$$\begin{aligned} \Delta U &= 0, \qquad \text{in} \quad \Omega, \\ U &= f, \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

- It seems unnecessary to worry about discretizing the entire $\Omega$ as is done in FEM or FD.
- As we will see later, rewriting Laplace's equation as an integral equation results in an integral equation on the boundary $\partial\Omega$ only.
- We reduce the dimension of the problem by one.
- If the right hand side is non-zero, however, things are different.

# Why integral equation methods?

- For some problems, the boundaries move. One example is bubbles and drops in fluid dynamics.
- It is much easier to keep track of the boundaries only, rather than say a FEM mesh convering the whole of the domain.



# Why integral equation methods?

- Sometimes it is natural to have an infinite domain, for example in electromagnetic scattering problems.
- We can't set up an infinite FEM or FD mesh, we need to introduce a wall somewhere.
- Since integral equations most often are on the boundaries only, there is no need for a wall.

# When do integral equation methods apply?

- We can only do linear problems.
- It is not necessary for the problem to be elliptic, but the ones treated with integral equation methods often are.
- If we want the reduction in dimension, we
  - cannot have non-zero right hand sides,
  - only have piecewise constant coefficients, for example in the heat conduction problem

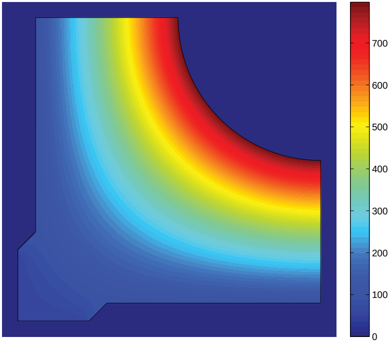
$$\nabla \cdot (\sigma \nabla T) = 0.$$

- So integral equation methods aren't very versatile, but when they apply they are very powerful.

# Examples : Laplace's equation

$$\Delta U = 0, \qquad \text{in} \quad \Omega$$

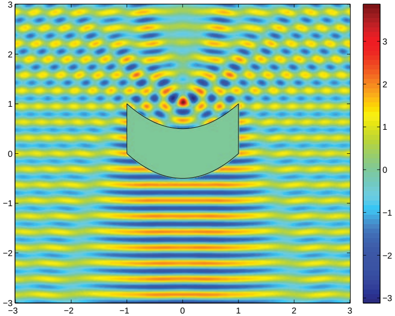
Heat conduction, electrostatics, etc.



# Examples : Helmholtz' equation

$$\Delta U + k^2 U = 0, \qquad \text{in} \quad \Omega$$

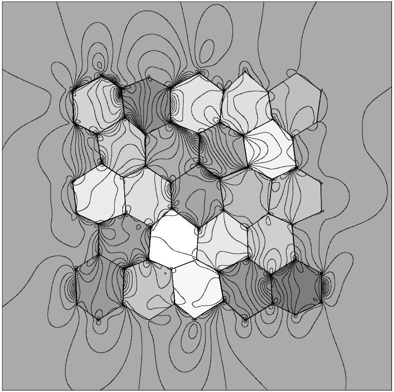
Electromagnetic scattering, acoustics.

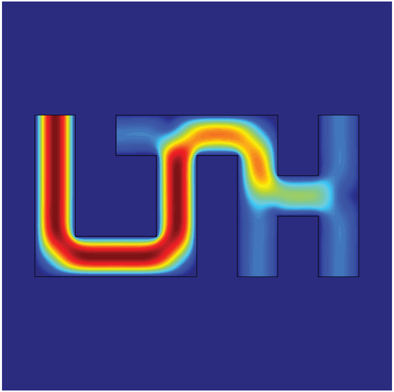


# Examples : The biharmonic equation

$$\Delta^2 U = 0, \qquad \text{in} \quad \Omega$$

Linear elasticity, Stokes flow.





# Integral equation formulations of PDEs

- What are we aiming for? What is an integral equation?
- We will encounter two types :

$$\int_{\partial\Omega} \rho(x) K(x,y)\,\mathrm{d}l_x = g(y), \qquad y \in \partial\Omega$$

$$\mu(y) + \int_{\partial\Omega} \mu(x) K(x,y)\,\mathrm{d}l_x = g(y), \qquad y \in \partial\Omega$$

- These are called Fredholm integral equations of the first and second kind, respectively.
- We call the known functions $K(x,y)$ and $g(y)$ the kernel and the right hand side. The unknown functions that we wish to compute are $\rho(x)$ and $\mu(x)$.
- These equations very seldom have closed form solutions. We need to solve them numerically.

# Integral equation formulation of Laplace's equation

- So, how do we turn a PDE into an integral equation?
- For simplicity, we will mainly look at Laplace's equation in 2D

$$\begin{aligned} \Delta U &= 0, \qquad \text{in} \quad \Omega, \\ U &= f, \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

- The Green's function $G(x, x_0)$ of the Laplace operator $\Delta$ satisfies

$$\Delta G(x, x_0) = \delta(x - x_0)$$

where $\delta(x - x_0)$ is the Dirac delta distribution at $x_0$. In physics, there is often an extra minus sign.

# Integral equation formulation of Laplace's equation

- Let us begin with Gauss' divergence theorem

$$\int_\Omega \nabla \cdot F \,\mathrm{d}A = \int_{\partial\Omega} F \cdot n \,\mathrm{d}l,$$

  where $n$ is the outward unit normal.

- If we set $F = v\nabla u - u\nabla v$ we get Green's second identity

$$\int_\Omega v\Delta u - u\Delta v \,\mathrm{d}A = \int_{\partial\Omega} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\,\mathrm{d}l$$

  which holds for $u, v \in C^2(\Omega)$.

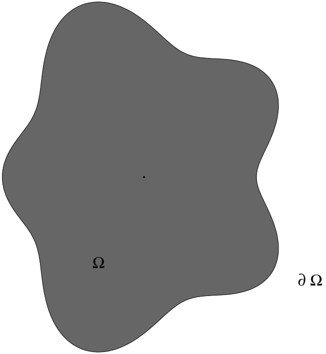
- Now we set $u = U$ and $v = G$.

# Integral equation formulation of Laplace's equation

- We "get"
$$U(x_0) = \int_{\partial\Omega} U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n}\, \mathrm{d}l$$
for any $x_0$ in $\Omega$.
- So, the solution to Laplace's equation inside $\Omega$ can be written as an integral over the boundary.
- But what is $G$ really? Is it really twice continuously differentiable?

# The Green's function

- The Laplace operator is scale, translation and rotation invariant.
- Assume that $x_0 = (0, 0)$ and that the domain $\Omega$ contains the origin.



# The Green's function

- We get:

$$1 = \int_\Omega \delta(x)\,\mathrm{d}A = \int_\Omega \Delta G\,\mathrm{d}A = \int_\Omega \nabla\cdot\nabla G\,\mathrm{d}A =$$
$$= \int_{\partial\Omega} \nabla G\cdot n\,\mathrm{d}l = \int_{\partial\Omega} \frac{\partial G}{\partial n}\,\mathrm{d}l$$

- Because of rotational invariance, we expect $G(x,0)$ to depend only on $r = |x|$.
- Take $\Omega_c$ to be a circle, centered at the origin and with radius $a$. On $\partial\Omega_c$ we then have

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r},$$

  and $\frac{\partial G}{\partial r} = \text{Const.}$ there.

# The Green's function

- We get:

$$1 = \int_{\partial\Omega_c} \frac{\partial G}{\partial n}\,\mathrm{d}l = \int_0^{2\pi} \frac{\partial G}{\partial r} a\,\mathrm{d}\theta = 2\pi a \frac{\partial G}{\partial r},$$

  which holds for any $a$.

- We pick $a = r$ and get

$$\frac{\partial G}{\partial r} = \frac{1}{2\pi r}$$

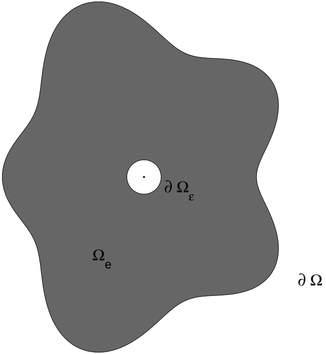
$$G(r) = \frac{1}{2\pi}\log(r) + C$$

- Setting $C = 0$ for physical reasons, we finally get

$$G(x, x_0) = \frac{1}{2\pi}\log(|x_0 - x|).$$

# Green's representation formula

- So, let us return to
$$\int_{\Omega} G\Delta U - U\Delta G \, \mathrm{d}A = \int_{\partial\Omega} G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n} \, \mathrm{d}l$$
and take a more thorough look.
- For simplicity, let $x_0 = 0$ and cut out a circle of radius $\varepsilon$ around the origin. On the circle, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$.



# Green's representation formula

- We now have two boundary components, but the left hand side above is zero on $\Omega_e$. We get

$$\begin{aligned}\int_{\partial\Omega} G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n}\,\mathrm{d}l &= \frac{1}{2\pi}\int_{\partial\Omega_\varepsilon} \log r\frac{\partial U}{\partial r} - U\frac{\partial}{\partial r}\log r\,\mathrm{d}l = \\ &= \frac{1}{2\pi}\left(2\pi\varepsilon\log\varepsilon\overline{\frac{\partial U}{\partial r}} - \frac{2\pi\varepsilon}{\varepsilon}\overline{U}\right)\end{aligned}$$

  where the bar denotes the mean over the circle $r = \varepsilon$.
- In the limit $\varepsilon \to 0$, the first term vanishes since the derivative of $U$ is bounded.
- The mean of $U$ over the circle goes to $U(0)$ as $\varepsilon \to 0$, so we get

$$U(0) = \int_{\partial\Omega} U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\,\mathrm{d}l$$

# Green's representation formula

- So it turned out that

$$U(x_0) = \int_{\partial\Omega} U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \, \mathrm{d}l \tag{1}$$

  for $x_0$ in $\Omega$ really was true. This formula is called Green's representation formula.

- But it does not help us, we need to know both $U$ and its normal derivative on $\partial\Omega$ to compute $U$ throughout $\Omega$. In general we don't.
- If (1) held on the boundary, however, since $U = f$ on the boundary, we'd be able to get a first kind integral equation on $\partial\Omega$ with $\frac{\partial U}{\partial n}$ as the unknown by substituting all $U$'s by the boundary data $f$.

$$\int_{\partial\Omega} G \frac{\partial U}{\partial n} \, \mathrm{d}l = \int_{\partial\Omega} f \frac{\partial G}{\partial n} \, \mathrm{d}l - f$$

# The jump relations

$$U(x_0) = \int_{\partial\Omega} U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \, \mathrm{d}l$$

- Sadly, this equation does not hold for $x_0$ on the boundary.
- For $x_0$ outside $\Omega$ the left hand side is zero as we saw with the cut out circle before, but just inside $\Omega$ it is not zero in general.
- The behavior of the integral above as $x_0$ approaches the boundary is given by the jump relations

$$\begin{aligned}
\lim_{x_0 \to \partial\Omega} \int_{\partial\Omega} G \frac{\partial U}{\partial n} \, \mathrm{d}l &= \int_{\partial\Omega} G \frac{\partial U}{\partial n} \, \mathrm{d}l \\
\lim_{\substack{x_0 \in \Omega \\ x_0 \to \partial\Omega}} \int_{\partial\Omega} U \frac{\partial G}{\partial n} \, \mathrm{d}l &= \frac{1}{2} U + \int_{\partial\Omega} U \frac{\partial G}{\partial n} \, \mathrm{d}l.
\end{aligned}$$

# Integral equation formulations for Laplace's equation

- So on the boundary $\partial\Omega$ we have the equation

$$\frac{1}{2}U(x_0) = \int_{\partial\Omega} U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\,\mathrm{d}l$$

- If $U = f$ is prescribed we get the first kind Fredholm integral equation

$$\int_{\partial\Omega} G\frac{\partial U}{\partial n}\,\mathrm{d}l = \int_{\partial\Omega} f\frac{\partial G}{\partial n}\,\mathrm{d}l - \frac{1}{2}f$$

- If $\frac{\partial U}{\partial n} = f$ we get the second kind equation

$$\frac{1}{2}U - \int_{\partial\Omega} U\frac{\partial G}{\partial n}\,\mathrm{d}l = -\int_{\partial\Omega} Gf\,\mathrm{d}l$$

but beware, this last equation is not uniquely solvable.

# A direct formulation

- We have finally reached a viable integral equation formulation of Laplace's equation. For our problem, with Dirichlet boundary conditions we do the following:
- Solve

$$\int_{\partial\Omega} G \frac{\partial U}{\partial n} \, \mathrm{d}l = \int_{\partial\Omega} f \frac{\partial G}{\partial n} \, \mathrm{d}l - \frac{1}{2} f$$

  for $\frac{\partial U}{\partial n}$.
- Knowing both $U$ and $\frac{\partial U}{\partial n}$ on the boundary we may compute the solution at any point $x_0$ inside the domain via

$$U(x_0) = \int_{\partial\Omega} U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \, \mathrm{d}l$$

- This is a two-stage process, unlike in FEM or FD.
- We call this a direct formulation, we are dealing with the solution $U$ and its derivative on the boundary directly.

# An indirect formulation

- We will not be using direct formulations here, however, but rather indirect ones.
- We call
$$U(x_0) = \frac{1}{2\pi} \int_{\partial\Omega} \rho(x) \log(|x_0 - x|) \, \mathrm{d}l(x)$$
a single layer potential or single layer representation of the solution $U$ inside $\Omega$, and
$$U(x_0) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(x) \frac{\partial}{\partial n} \log(|x_0 - x|) \, \mathrm{d}l(x)$$
a double layer potential or double layer representation.
- The terminology comes from potential theory where $\rho(x)$ and $\mu(x)$ are charge and dipole densities, respectively.
- Each of these can be used to represent any solution to Laplace's equation inside $\Omega$. We don't need both together as above.

# An indirect formulation

- You will be using the double layer representation in the homework.

$$U(x_0) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(x) \frac{\partial}{\partial n} \log(|x_0 - x|) \, \mathrm{d}l(x) \qquad (2)$$

- Using the appropriate jump relation we get together with the fact that $U(x_0) = f(x_0$ on the boundary that

$$\frac{1}{2}\mu(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \mu(x) \frac{\partial}{\partial n} \log(|x_0 - x|) \, \mathrm{d}l(x) = f(x_0)$$

holds on $\partial\Omega$. Once we have solved this equation for $\mu(x_0)$ we use (2) to compute the solution in the domain.

# Complex variables

- Since we are working in 2D, it is very natural to use complex variables. In fact, one can derive the same double layer representation as above using the theory of analytic functions.
- We will not derive the complex equation here, but is in fact easier to work with the complex variant. The integral equation is

$$\frac{1}{2}\mu(z_0) + \frac{1}{2\pi}\int_{\partial\Omega}\mu(z)\mathrm{Im}\left\{\frac{\mathrm{d}z}{z-z_0}\right\} = f(z_0),$$

  where Im denotes the imaginary part. This is a lot more compact than its real counterpart if you write that out.
- Having solved the above equation for $\mu$, the solution $U$ at a point in $\Omega$ is then computed via

$$U(z_0) = \frac{1}{2\pi}\int_{\partial\Omega}\mu(z)\mathrm{Im}\left\{\frac{\mathrm{d}z}{z-z_0}\right\}.$$

# Complex variables

- You will use the complex formulations in the homework. It will potentially spare you a lot of bug hunting.
- In order to do this you need to evaluate complex curve integrals on $\partial\Omega$.
- To do this you need a parameterization $z(t)$ of the boundary curve. For a circle, for example, we have $z(t) = e^{\mathrm{i}t}$ for $0 \leq t \leq 2\pi$.
- To evaluate an integral over the curve described by $z(t)$ we now get

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = \int_0^{2\pi} f(z(t))z'(t)\,\mathrm{d}t$$

- In the same way,

$$\frac{1}{2\pi}\int_{\partial\Omega} \mu(z)\mathrm{Im}\left\{\frac{\mathrm{d}z}{z - z_0}\right\} = \frac{1}{2\pi}\int_0^{2\pi} \mu(z(t))\mathrm{Im}\left\{\frac{z'(t)}{z(t) - z_0}\right\}\mathrm{d}t.$$