Boundary integral equation methods for elliptic problems - part 3

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So far...

- We have discussed Laplace's equation with Dirichlet boundary conditions

$$\begin{aligned} \Delta U &= 0, \qquad \text{in} \quad \Omega, \\ U &= f, \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

- One possible integral equation formulation of this problem is the double layer representation, where we represent the solution U as

$$U(z_0) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\}.$$

and solve the integral equation

$$\frac{1}{2}\mu(z_0) + \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\} = f(z_0),$$

for the unknown density μ.

So far...

- We found out that our integral operator was compact, and in turn that gave us
  - a way to prove existence and uniqueness of solutions to the integral equation via Fredholm's alternative,
  - a theoretical explanation why we could expect the iterative solver GMRES to work well on the discretized equation.
- We used Nyström discretization to get a linear system of equation to solve. Nyström discretization had the property that it preserves the order of the underlying quadrature rule in the solution of the integral equation.
- This was very good for us, since we found that the trapezoidal rule was very accurate for smooth and periodic integrands.
- Finally, we discussed a way to speed up the solution of the equation system significantly: the fast multipole method.

So, what is left?

- What about Laplace's equation with Neumann boundary conditions? Or mixed boundary conditions?
- What about the solution close to the boundary?
- What about non-smooth domains?
- What about other PDE:s?

Neumann boundary conditions

- The solution to

$$\begin{aligned} \Delta U &= 0, \qquad \text{in} \quad \Omega, \\ \frac{\partial U}{\partial n} &= g, \qquad \text{on} \quad \partial\Omega, \end{aligned}$$

  is not unique.
- We could set up an integral equation as before, though, but it makes better sense to look at

$$\begin{aligned} \Delta U &= 0, \qquad \text{in} \quad \Omega^c, \\ \frac{\partial U}{\partial n} &= g, \qquad \text{on} \quad \partial\Omega, \end{aligned}$$

  instead, where Ω^c is the complement of Ω.
- It is reasonable to look for solutions that decay to zero far away. This is equivalent to the condition

$$\int_{\partial\Omega} g \, \mathrm{d}l = 0.$$

Neumann boundary conditions

- So let us as before use the double layer representation

$$U(z_0) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\}.$$

- We want the normal derivative on $\partial\Omega$, so we compute

$$\frac{\partial U}{\partial n} = \frac{\partial}{\partial n} \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\},$$

  let z_0 approach the boundary and set this equal to g there. Just as before, right?

- Well, no. The normal derivative makes the kernel look like $\frac{1}{(z-z_0)^2}$.
- We call integral operators with such kernels hypersingular integral operators and they are a nasty bunch.

Neumann boundary conditions

- Instead we use a single layer representation, we touched on these briefly in the first lecture. Represent the solution as

$$U(x_0) = \int_{\partial\Omega} \rho(x) G(x, x_0)\, \mathrm{d}l,$$

or in complex variables

$$U(z_0) = \frac{1}{2\pi} \int_{\partial\Omega} \rho(z) \log(|z_0 - z|)\, |\mathrm{d}z|.$$

- Taking the normal derivative of this is no problem, we end up with something very similar to the double layer representation. However, the point where the normal is evaluated differs.
- We get the integral equation

$$\frac{1}{2}\rho(z_0) - \frac{1}{2\pi} \int_{\partial\Omega} \rho(z) \mathrm{Im} \left\{ \frac{n_{z_0} \overline{n_z} \mathrm{d}z}{z - z_0} \right\} = g(z_0),$$

for the Neumann problem. Same properties as for Dirichlet.

Mixed boundary conditions

- A common problem is

$$\begin{aligned} \Delta U &= 0, && \text{in} \quad \Omega, \\ U &= f, && \text{on} \quad \partial\Omega_D, \\ \frac{\partial U}{\partial n} &= g, && \text{on} \quad \partial\Omega_N. \end{aligned}$$

- Drawing on past experiences we represent the solution as

$$U(z_0) = \frac{1}{2\pi} \int_{\partial\Omega_D} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\} + \frac{1}{2\pi} \int_{\partial\Omega_N} \mu(z) \log(|z_0 - z|) \, |\mathrm{d}z|.$$

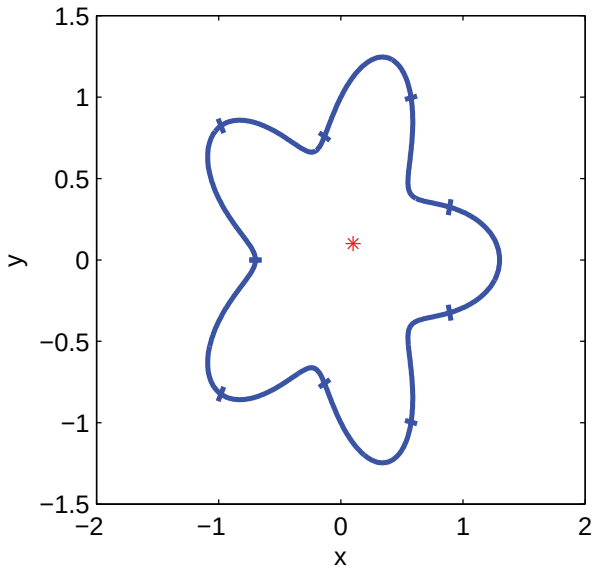
- This works, and we get a system of integral equations for the mixed density μ. We could also do Robin boundary conditions in this way.
- But what happens where different boundary conditions meet? Bad things, actually pretty much the same things as with the corners of non-smooth domains. We will look at these later.

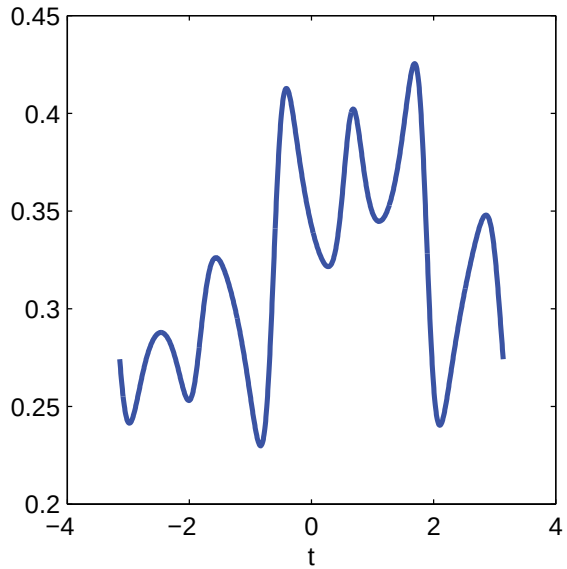
The solution close to the boundary

- Let us return to Laplace's equation with Dirichlet boundary conditions.
- Assume we have solved the integral equation and have obtained the density μ. Now we wish to compute the solution in the domain using

$$U(z_0) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\}.$$

- We do OK in the center of the domain.





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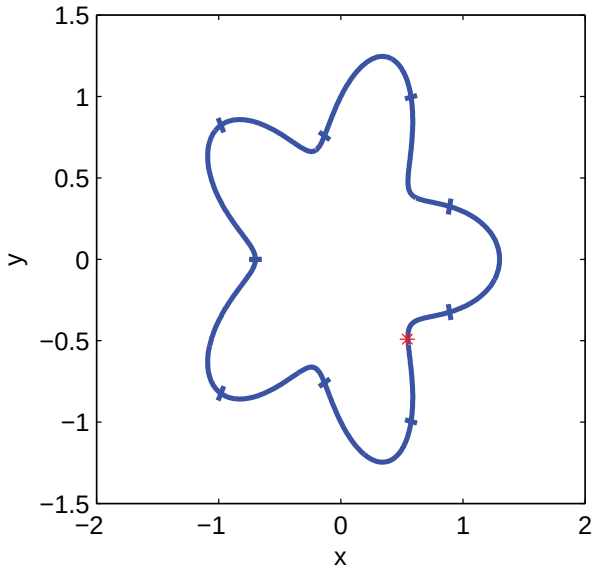
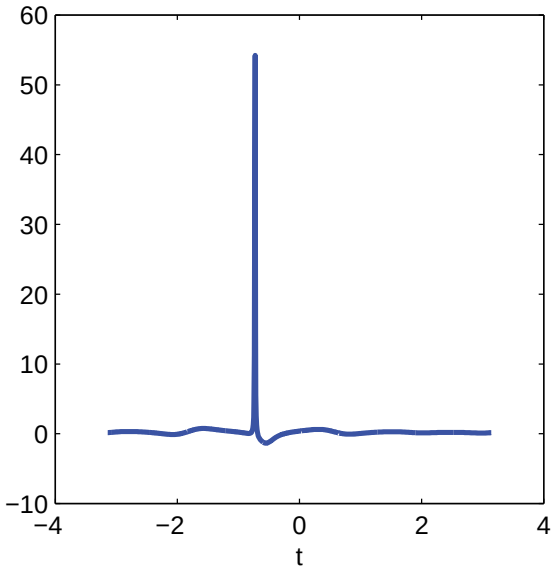
$$U(z_0) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\}.$$

- But not close to the boundary.



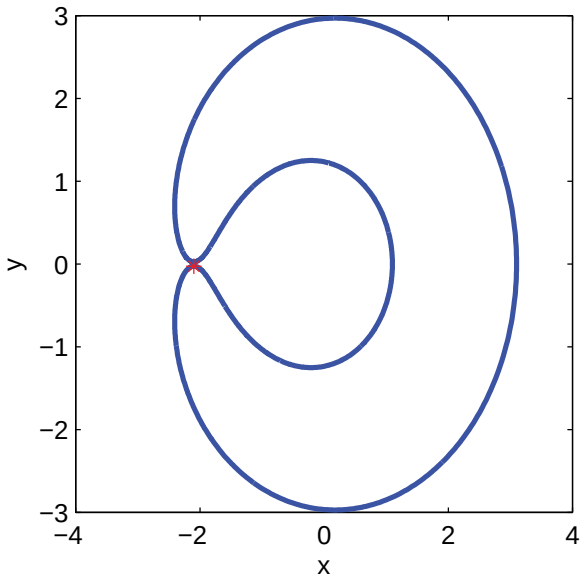


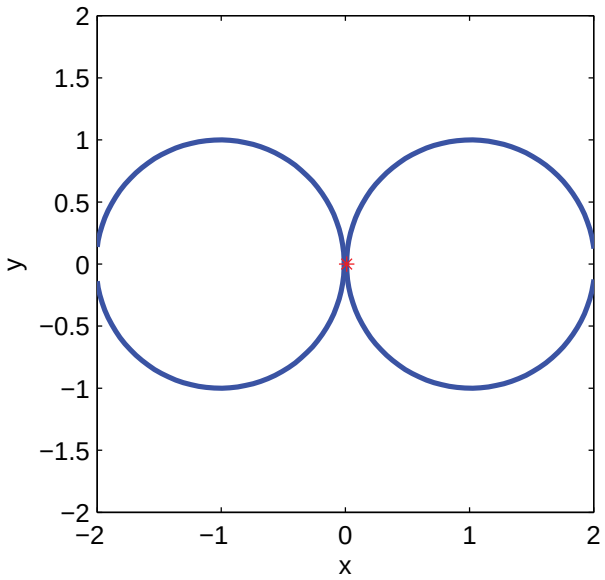
The solution close to the boundary

- The same problem occurs when we want to solve the integral equation

$$\frac{1}{2}\mu(z_0) + \frac{1}{2\pi}\int_{\partial\Omega}\mu(z)\mathrm{Im}\left\{\frac{\mathrm{d}z}{z-z_0}\right\} = f(z_0),$$

for domains whose boundary falls back on itself.





The solution close to the boundary

- So let us see what we can do if we wish to evaluate the solution close to the boundary. We will restrict ourselves to the boundary segment closest to the point, since that is where the integrand is troublesome.
- We note that it is the behavior of $\frac{1}{z-z_0}$ that is the problem, not μ which is in fact well-behaved in most cases.
- Since the density μ is real we may remove the imaginary part temporarily. That is, we want to compute

$$\tilde{U}(z_0) = \frac{1}{2\pi} \int_{\partial\Omega_{\text{part}}} \mu(z) \frac{\mathrm{d}z}{z - z_0}.$$

The solution close to the boundary

- So, $\mu(z)$ is well-behaved. Then let us expand $\mu(z)$ as an nth degree polynomial in z over the boundary segment.

$$\mu(z) \approx \sum_{k=0}^{n} c_k z^k$$

- Substituting, we get

$$\tilde{U}(z_0) = \frac{1}{2\pi} \int_{\partial\Omega_{\text{part}}} \mu(z) \frac{\mathrm{d}z}{z - z_0} \approx \frac{1}{2\pi} \sum_{k=0}^{n} c_k \int_{\partial\Omega_{\text{part}}} \frac{z^k \mathrm{d}z}{z - z_0}.$$

- These last integrals, which are not easily evaluated numerically, we can evaluate analytically. The theory of analytic function tells us that the curve integral of an analytic function is independent of the path. We may choose a straight line.
- But we must be careful not to pass z_0 when we deform our path. The integrands are singular there, and hence not analytic.

The solution close to the boundary

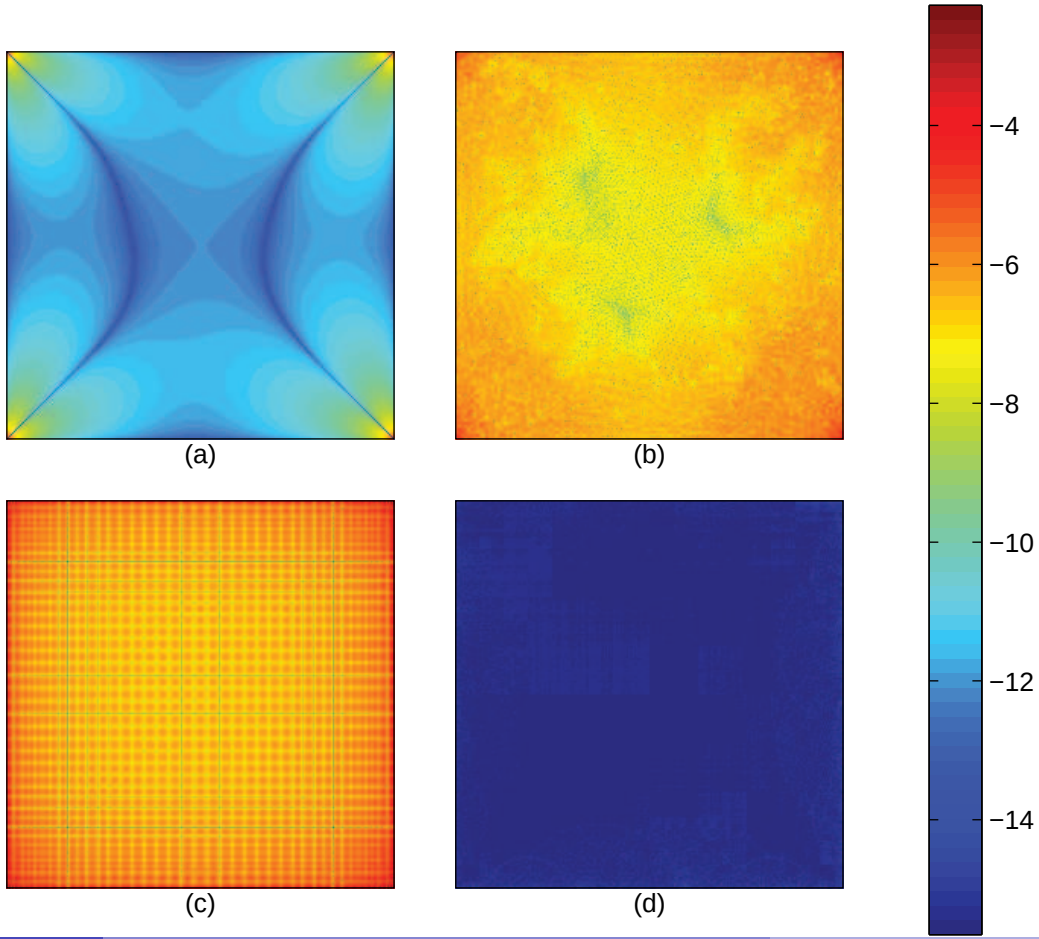
- So this looks nice, but haven't we overlooked something?
- We compute c_k such that

$$\mu(z) \approx \sum_{k=0}^{n} c_k z^k,$$

  which is basically interpolation. And our points are equispaced.
- High-order interpolation for equispaced points is a bad idea (Runge's phenomenon).
- So we throw away the trapezoidal rule and instead go for Gaussian quadrature. The points are now better spaced for interpolation, they are in fact "almost" Chebyshev nodes which are optimal for interpolation.
- We lose the amazing accuracy of the trapezoidal rule integration, but if we use, say, a 16-point Gaussian rule we get 32nd order, which is not bad either. Now, however, we have no problems close to the boundary.

Non-smooth domains

- Non-smooth, or piece-wise smooth domains are a headache for integral equation solvers.
- In fact, they are a problem for basically any solution strategy. Below, (a) FD, (b) FEM, (c) Spectral method, (d) BIE



Non-smooth domains

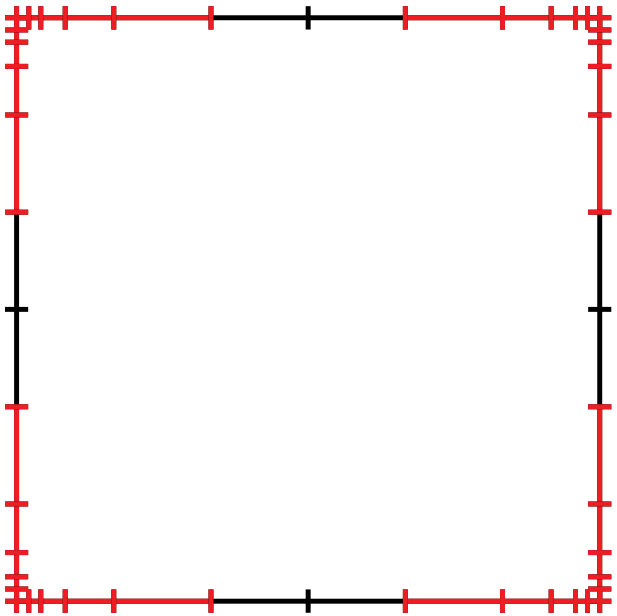
- For integral equation solvers the main problem is that the compactness of the integral operator is lost. The kernel is singular at the corners.
- As a result we lose Fredholm's alternative and the nice spectral properties for GMRES.
- Refining the discretization close to corners is a common strategy, but more points equals more work and we cannot be sure the number of GMRES iterations stays bounded anymore either.
- We can refine the discretization while retaining the nice GMRES convergence by constructing a preconditioner.

Non-smooth domains

- Let us say we have discretized the integral equation on a refined mesh with Gaussian quadrature. The resulting matrix is $\mathbf{K}$, and we have the equation system

$$(\frac{1}{2}\mathbf{I} + \mathbf{K})\boldsymbol{\mu} = \mathbf{f}.$$

- We do the split $\mathbf{K} = \mathbf{K}^{\circ} + \mathbf{K}^{\star}$ where $\mathbf{K}^{\star}$ contains the elements corresponding to self interaction at the corners. The red portions below. All the bad behavior is now in $\mathbf{K}^{\star}$.



Non-smooth domains

- We get

$$(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\circ} + \mathbf{K}^{\star})\boldsymbol{\mu} = \mathbf{f}.$$

$$(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\circ} + \mathbf{K}^{\star})(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\star})^{-1}(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\star})\boldsymbol{\mu} = \mathbf{f}.$$

- And setting $\tilde{\boldsymbol{\mu}} = (\frac{1}{2}\mathbf{I} + \mathbf{K}^{\star})\boldsymbol{\mu}$ we get

$$\left(\mathbf{I} + \mathbf{K}^{\circ}(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\star})^{-1}\right)\tilde{\boldsymbol{\mu}} = \mathbf{f}.$$

- It turns out that we only need the refined mesh for computing $(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\star})^{-1}$, we can solve the main equation on an unrefined mesh, saving work.
- Also, $(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\star})^{-1}$ can be computed very rapidly using a recursion.
- This strategy also works for the interfaces in mixed boundary condition problems.

A Laplace demo

- So using this corner strategy and the scheme for computing the solution close to the boundary, how well does integral equation methods do for Laplace's equation?

Other PDEs

- The derivation of boundary integral equation formulations for other PDEs are often similar to what we have done for Laplace's equation.
- One often starts with a fundamental solution, or Green's function, for the problem and use it to represent the solution of the specific problem at hand.
- For Helmholtz' equation

$$\Delta U + k^2 U = 0, \qquad \text{in} \quad \Omega,$$

the Green's function is

$$G(z, z_0) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|z - z_0|)$$

where $H_0^{(1)}(z)$ is the zeroth order Hankel function of the first kind. It is related to the Bessel functions and is singular and oscillatory.

- One uses a sum of a single and double layer representation to represent the solution.

Other PDEs

- For Stokes flow, the Green's function is

$$G(z, z_0) = \frac{1}{2} - \log(|z - z_0|) + \frac{1}{2}\frac{z - z_0}{\bar{z} - \overline{z_0}}\mathcal{C}$$

where bar denotes complex conjugation and $\mathcal{C}$ is the complex conjugation operator.