Homework 5 : Boundary integral equation methods for elliptic problems

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In this homework you will solve Laplace's equation with Dirichlet boundary conditions on two dimensional domains. There will be both theoretical and programming problems.

The problem we wish to solve is

$$\Delta U(x) = 0, \qquad x \in \Omega, \tag{1}$$
$$U(x) = f(x), \qquad x \in \partial\Omega, \tag{2}$$

where Ω is a smooth domain in 2D with boundary $\partial\Omega$.

We begin by expressing the solution U as a double layer potential,

$$U(x_0) = \int_{\partial\Omega} \frac{\partial G(x, x_0)}{\partial n_x} \mu(x) \mathrm{d}l, \qquad x_0 \in \Omega \tag{3}$$

with n_x being the outward unit normal at x and $G(x, x_0) = \frac{1}{2\pi} \log(|x_0 - x|)$ is the Green's function for the Laplace operator in the plane. The integral is with respect to arclength. To compute the solution at some point x_0 we need to first compute the unknown double layer density μ. As x_0 approaches the boundary in (3), the jump relations together with the boundary conditions give the integral equation

$$\frac{1}{2}\mu(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \mu(x) \frac{\partial}{\partial n_x} \log(|x_0 - x|) \, \mathrm{d}l = f(x_0), \qquad x_0 \in \partial\Omega \tag{4}$$

for μ. It is common to absorb the factor 1/2 into the density μ, in which case (3) must be adjusted accordingly.

In complex notation the same equation is

$$\frac{1}{2}\mu(z_0) + \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\} = f(z_0), \qquad z_0 \in \partial\Omega, \tag{5}$$

where Im denotes the imaginary part. Note that the density is still real. When (5) has been solved for μ the solution in Ω can be computed via

$$U(z_0) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(z) \mathrm{Im} \left\{ \frac{\mathrm{d}z}{z - z_0} \right\}, \qquad z_0 \in \Omega. \tag{6}$$

1) : Confirm that the real and complex equations indeed are equivalent, for example by parameterizing both equations and comparing the results.

We will use complex representations here. The equations generally turn out more compact that way, and simpler to program. Furthermore, it is easier to handle your next task in the complex plane. Suppose that the boundary $\partial\Omega$ has the parameterization $z(t)$ for $t \in [a, b]$. The integral equation (5) then becomes

$$\frac{1}{2}\mu(z(s)) + \frac{1}{2\pi}\int_a^b \mu(z(t))\mathrm{Im}\left\{\frac{z'(t)}{z(t)-z(s)}\right\}\mathrm{d}t = f(z(s)), \qquad s \in [a,b], \quad (7)$$

and it seems that we may have a problem for $t = s$, that there is a singularity in the integrand there. It turns out that we are fine, though. For smooth $z(t)$, the limit

$$\lim_{h\to 0}\mathrm{Im}\left\{\frac{z'(t)}{z(t)-z(t+h)}\right\} \quad (8)$$

exists.

2) : Compute the limit (8) in terms of derivatives of $z(t)$. Use Taylor expansions and remember that you are dealing with complex valued functions.

To solve (7) we need to evaluate integrals numerically, and we will use the trapezoidal rule. For periodic and smooth integrands the trapezoidal rule has very nice convergence properties. Using Nyström discretization/method, you are now in a position to solve Laplace's equation. As for what domain to use, any smooth, reasonably well behaved one can be tried. Circles and such may be too boring, though, and a parametrization for a somewhat more interesting five-armed starfish domain is

$$z(t) = (1 + 0.3\cos(5t))\, e^{it}, \qquad t \in [0, 2\pi] \quad (9)$$

Furthermore, we would like to choose the boundary conditions in such a way that we know the correct solution for comparisons. For this we may use

$$f(z) = \mathrm{Im}\left\{\frac{1}{z - z_p}\right\}, \qquad z \in \partial\Omega, \quad (10)$$

where z_p is not in Ω. As a function of two real variables, if $z = x + \mathrm{i}y$ and $z_p = x_p + \mathrm{i}y_p$, the function f is

$$f(x,y) = -\frac{y - y_p}{(x-x_p)^2 + (y-y_p)^2}. \quad (11)$$

3) : If (10) is used as right hand side what will $U(z), z \in \Omega$ be and why?

4) : Use the parameterization of the five-armed starfish domain, and use Nyström discretization to set up a linear equation system for solving (5) with the right hand side as in 10, not forgetting the diagonal elements you computed the limits for. Solve it using Gaussian elimination and use the discrete density to compute the solution at points along a line from the center of the starfish to the tip of one of its arms. Compare with the known solution. What happens and why? Try different numbers of discretization points.

5) : Compute the condition number of the system matrix for increasing numbers of discretization points and plot them. What is the difference in behaviour compared to finite difference schemes? Also, plot the eigenvalues of the matrix for increasing numbers of discretization points. What does this suggest regarding the nature of the integral operator?

6) : Solve the linear system to full accuracy using Matlab's in-built iterative solver `gmres` for increasing numbers of discretization points. What happens in terms of iterations required and what does this mean for the asymptotic time complexity of the solver? What would happen in this regard if the fast multipole method was employed?