Let us solve for $\hat{\lambda}_i$ and $\hat{z}_i(x)$: It is easy to see that $\hat{z}_i(x)$ must equal $\alpha\sin(\sqrt{\hat{\lambda}_i}x)+\beta\cos(\sqrt{\hat{\lambda}_i}x)$ for some constants α and β. The boundary condition $\hat{z}_i(0)=0$ implies $\beta=0$, and the boundary condition $\hat{z}_i(1)=0$ implies that $\sqrt{\hat{\lambda}_i}$ is an integer multiple of π, which we can take to be $i\pi$. Thus $\hat{\lambda}_i=i^2\pi^2$ and $\hat{z}_i(x)=\alpha\sin(i\pi x)$ for any nonzero constant α (which we can set to 1). Thus the eigenvector z_i is *precisely* equal to the eigenfunction $\hat{z}_i(x)$ evaluated at the sample points $x_j=jh$ (when scaled by $\sqrt{\frac{2}{N+1}}$). And when i is small, $\hat{\lambda}_i=i^2\pi^2$ is well approximated by $h^{-2}\cdot\lambda_i=(N+1)^2\cdot 2(1-\cos\frac{i\pi}{N+1})=i^2\pi^2+O((N+1)^{-2})$.

Thus we see there is a close correspondence between T_N (or $h^{-2}T_N$) and the second derivative operator $-\frac{d^2}{dx^2}$. This correspondence will be the motivation for the design and analysis of later algorithms.

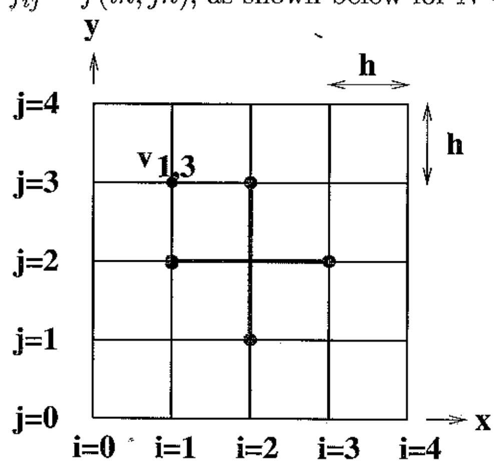
It is also possible to write down simple formulas for the Cholesky and LU factors of T_N; see Question 6.2 for details.

6.3.2. Poisson's Equation in Two Dimensions

Now we turn to Poisson's equation in two dimensions:

$$-\frac{\partial^2 v(x,y)}{\partial x^2}-\frac{\partial^2 v(x,y)}{\partial y^2}=f(x,y) \tag{6.6}$$

on the unit square $\{(x,y):\ 0<x,y<1\}$, with boundary condition $v=0$ on the boundary of the square. We discretize at the grid points in the square which are at (x_i,y_j) with $x_i=ih$ and $y_j=jh$, with $h=\frac{1}{N+1}$. We abbreviate $v_{ij}=v(ih,jh)$ and $f_{ij}=f(ih,jh)$, as shown below for $N=3$:



From equation (6.2), we know that we can approximate

$$-\left.\frac{\partial^2 v(x,y)}{\partial x^2}\right|_{x=x_i,y=y_j}\approx\frac{2v_{i,j}-v_{i-1,j}-v_{i+1,j}}{h^2}\quad\text{and} \tag{6.7}$$

$$-\left.\frac{\partial^2 v(x,y)}{\partial y^2}\right|_{x=x_i,y=y_j}\approx\frac{2v_{i,j}-v_{i,j-1}-v_{i,j+1}}{h^2}. \tag{6.8}$$

Adding these approximations lets us write

$$-\frac{\partial^2 v(x,y)}{\partial x^2} - \frac{\partial^2 v(x,y)}{\partial y^2}\bigg|_{x=x_i,y=y_j}$$
$$= \frac{4v_{ij} - v_{i-1,j} - v_{i+1,j} - v_{i,j-1} - v_{i,j+1}}{h^2} - \tau_{ij}, \tag{6.9}$$

where τ_{ij} is again a truncation error bounded by $O(h^2)$. The heavy (blue) cross in the middle of the above figure is called the (*5-point*) *stencil* of this equation, because it connects all (5) values of v present in equation (6.9). From the boundary conditions we know $v_{0j} = v_{N+1,j} = v_{i,0} = v_{i,N+1} = 0$ so that equation (6.9) defines a set of $n = N^2$ linear equations in the n unknowns v_{ij} for $1 \le i, j \le N$:

$$4v_{ij} - v_{i-1,j} - v_{i+1,j} - v_{i,j-1} - v_{i,j+1} = h^2 f_{ij}. \tag{6.10}$$

There are two ways to rewrite the n equations represented by (6.10) as a single matrix equation, both of which we will use later.

The first way is to think of the unknowns v_{ij} as occupying an N-by-N matrix V with entries v_{ij} and the right-hand sides $h^2 f_{ij}$ as similarly occupying an N-by-N matrix $h^2 F$. The trick is to write the matrix with i,j entry $4v_{ij} - v_{i-1,j} - v_{i+1,j} - v_{i,j-1} - v_{i,j+1}$ in a simple way in terms of V and T_N: Simply note that

$$\begin{aligned} 2v_{ij} - v_{i-1,j} - v_{i+1,j} &= (T_N \cdot V)_{ij}, \\ 2v_{ij} - v_{i,j-1} - v_{i,j+1} &= (V \cdot T_N)_{ij}, \end{aligned}$$

so adding these two equations yields

$$(T_N \cdot V + V \cdot T_N)_{ij} = 4v_{ij} - v_{i-1,j} - v_{i+1,j} - v_{i,j-1} - v_{i,j+1} = h^2 f_{ij} = (h^2 F)_{ij}$$

or

$$T_N \cdot V + V \cdot T_N = h^2 F. \tag{6.11}$$

This is a linear system of equations for the unknown entries of the matrix V, even though it is not written in the usual "$Ax = b$" format, with the unknowns forming a vector x. (We will write the "$Ax = b$" format below.) Still, it is enough to tell us what the eigenvalues and eigenvectors of the underlying matrix A are, because "$Ax = \lambda x$" is the same as "$T_N V + V T_N = \lambda V$." Now suppose that $T_N z_i = \lambda_i z_i$ and $T_N z_j = \lambda_j z_j$ are any two eigenpairs of T_N, and let $V = z_i z_j^T$. Then

$$\begin{aligned} T_N V + V T_N &= (T_N z_i) z_j^T + z_i (z_j^T T_N) \\ &= (\lambda_i z_i) z_j^T + z_i (z_j^T \lambda_j) \\ &= (\lambda_i + \lambda_j) z_i z_j^T \\ &= (\lambda_i + \lambda_j) V, \end{aligned} \tag{6.12}$$

so $V = z_i z_j^T$ is an "eigenvector" and $\lambda_i + \lambda_j$ is an eigenvalue. Since V has N^2 entries, we expect N^2 eigenvalues and eigenvectors, one for each pair of eigenvalues λ_i and λ_j of T_N. In particular, the smallest eigenvalue is $2\lambda_1$ and the largest eigenvalue is $2\lambda_N$, so the condition number is the same as in the one-dimensional case. We rederive this result below using the "$Ax = b$" format. See Figure 6.3 for plots of some eigenvectors, represented as surfaces defined by the matrix entries of $z_i z_j^T$.

Just as the eigenvalues and eigenvectors of $h^{-2}T_N$ were good approximations to the eigenvalues and eigenfunctions of one-dimensional Poisson's equation, the same is true of two-dimensional Poisson's equation, whose eigenvalues and eigenfunctions are as follows (see Question 6.3):

$$\begin{aligned}\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) & \sin(i\pi x)\sin(j\pi y) \\ &= (i^2\pi^2 + j^2\pi^2)\sin(i\pi x)\sin(j\pi y). \end{aligned} \tag{6.13}$$

The second way to write the n equations represented by equation (6.10) as a single matrix equation is to write the unknowns v_{ij} in a single long N^2-by-1 vector. This requires us to choose an order for them, and we (somewhat arbitrarily) choose to number them as shown in Figure 6.4, columnwise from the upper left to the lower right.

For example, when $N = 3$ one gets a column vector $v \equiv [v_1, \ldots, v_9]^T$. If we number f accordingly, we can transform equation (6.10) to get

$$\begin{aligned} T_{3\times 3}\cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ \vdots \\ v_9 \end{bmatrix} &\equiv \left[\begin{array}{ccc|ccc|ccc} 4 & -1 & & -1 & & & & & \\ -1 & 4 & -1 & & -1 & & & & \\ & -1 & 4 & & & -1 & & & \\ \hline -1 & & & 4 & -1 & & -1 & & \\ & -1 & & -1 & 4 & -1 & & -1 & \\ & & -1 & & -1 & 4 & & & -1 \\ \hline & & & -1 & & & 4 & -1 & \\ & & & & -1 & & -1 & 4 & -1 \\ & & & & & -1 & & -1 & 4 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ \vdots \\ v_9 \end{bmatrix} \\ &= h^2 \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_9 \end{bmatrix}. \end{aligned} \tag{6.14}$$

The -1's immediately next to the diagonal correspond to subtracting the top and bottom neighbors $-v_{i,j-1} - v_{i,j+1}$. The -1's farther away away from the diagonal correspond to subtracting the left and right neighbors $-v_{i-1,j} - v_{i+1,j}$. For general N, we confirm in the next section that we get an N^2-by-N^2 linear system

$$T_{N\times N}\cdot v = h^2 f, \tag{6.15}$$

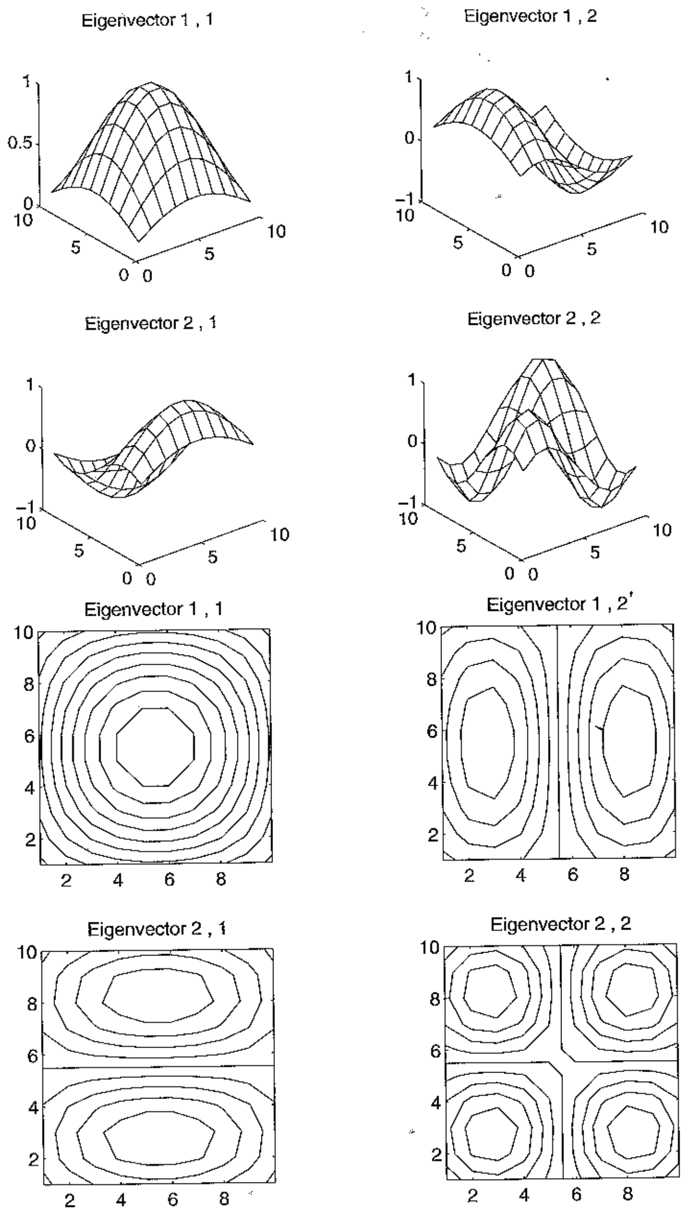


Fig. 6.3. *Three-dimensional and contour plots of first four eigenvectors of the* 10*-by-*10 *Poisson equation.*

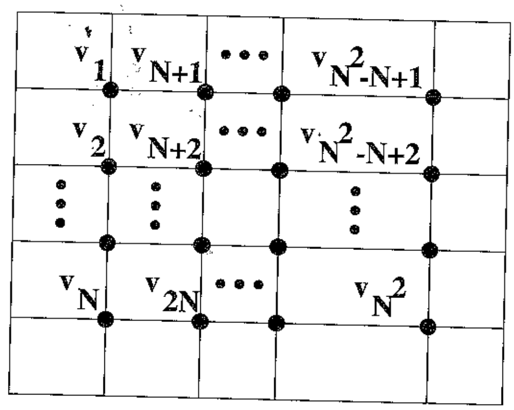
V = 

Fig. 6.4. *Numbering the unknowns in Poisson's equation.*

where $T_{N\times N}$ has N N-by-N blocks of the form $T_N + 2I_N$ on its diagonal and $-I_N$ blocks on its offdiagonals:

$$T_{N\times N} = \begin{bmatrix} T_N + 2I_N & -I_N & & \\ -I_N & \ddots & \ddots & \\ & \ddots & \ddots & -I_N \\ & & -I_N & T_N + 2I_N \end{bmatrix}. \quad (6.16)$$

6.3.3. Expressing Poisson's Equation with Kronecker Products

Here is a systematic way to derive equations (6.15) and (6.16) as well as to compute the eigenvalues and eigenvectors of $T_{N\times N}$. The method works equally well for Poisson's equation in three or more dimensions.

DEFINITION 6.1. *Let X be m-by-n. Then* $\text{vec}(X)$ *is defined to be a column vector of size $m \cdot n$ made of the columns of X stacked atop one another from left to right.*

Note that N^2-by-1 vector v defined in Figure 6.4 can also be written $v = \text{vec}(V)$.

To express $T_{N\times N}$ as well as compute its eigenvalues and eigenvectors, we need to introduce *Kronecker products.*

DEFINITION 6.2. *Let A be an m-by-n matrix and B be a p-by-q matrix. Then $A \otimes B$, the* Kronecker product *of A and B, is the $(m\cdot p)$-by-$(n \cdot q)$ matrix*

$$\begin{bmatrix} a_{1,1}\cdot B & \dots & a_{1,n}\cdot B \\ \vdots & & \vdots \\ a_{m,1}\cdot B & \dots & a_{m,n}\cdot B \end{bmatrix}.$$

The following lemma tells us how to rewrite the Poisson equation in terms of Kronecker products and the vec($\cdot$) operator.

LEMMA 6.2. *Let A be m-by-m, B be n-by-n, and X and C be m-by-n. Then the following properties hold:*

1. $\mathrm{vec}(AX) = (I_n \otimes A) \cdot \mathrm{vec}(X)$.
2. $\mathrm{vec}(XB) = (B^T \otimes I_m) \cdot \mathrm{vec}(X)$.
3. *The Poisson equation* $T_N V + V T_N = h^2 F$ *is equivalent to*

$$T_{N\times N} \cdot \mathrm{vec}(V) \equiv (I_N \otimes T_N + T_N \otimes I_N) \cdot \mathrm{vec}(V) = h^2 \mathrm{vec}(F). \quad (6.17)$$

Proof. We prove only part 3, leaving the other parts to Question 6.4. We start with the Poisson equation $T_N V + V T_N = h^2 F$ as expressed in equation (6.11), which is clearly equivalent to

$$\mathrm{vec}(T_N V + V T_N) = \mathrm{vec}(T_N V) + \mathrm{vec}(V T_N) = \mathrm{vec}(h^2 F).$$

By part 1 of the lemma

$$\mathrm{vec}(T_N V) = (I_N \otimes T_N)\mathrm{vec}(V).$$

By part 2 of the lemma and the symmetry of T_N,

$$\mathrm{vec}(V T_N) = (T_N^T \otimes I_N)\mathrm{vec}(V) = (T_N \otimes I_N)\mathrm{vec}(V).$$

Adding the last two expressions completes the proof of part 3. □

The reader can confirm that the expression

$$\begin{aligned} T_{N\times N} &= I_N \otimes T_N + T_N \otimes I_N \\ &= \begin{bmatrix} T_N & & & \\ & \ddots & & \\ & & \ddots & \\ & & & T_N \end{bmatrix} + \begin{bmatrix} 2I_N & -I_N & & \\ -I_N & \ddots & \ddots & \\ & \ddots & \ddots & -I_N \\ & & -I_N & 2I_N \end{bmatrix} \end{aligned}$$

from equation (6.17) agrees with equation (6.16).[26]

To compute the eigenvalues of matrices defined by Kronecker products, like $T_{N\times N}$, we need the following lemma, whose proof is also part of Question 6.4.

LEMMA 6.3. *The following facts about Kronecker products hold:*

1. *Assume that the products $A \cdot C$ and $B \cdot D$ are well defined. Then* $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$.

[26] We can use this formula to compute $T_{N\times N}$ in two lines of Matlab:

```
TN = 2*eye(N) - diag(ones(N-1,1),1) - diag(ones(N-1,1),-1);
TNxN = kron(eye(N),TN) + kron(TN,eye(N));
```

2. *If A and B are invertible, then* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

3. $(A \otimes B)^T = A^T \otimes B^T$.

PROPOSITION 6.1. *Let* $T_N = Z\Lambda Z^T$ *be the eigendecomposition of* T_N, *with* $Z = [z_1, \ldots, z_N]$ *the orthogonal matrix whose columns are eigenvectors, and* $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. *Then the eigendecomposition of* $T_{N\times N} = I \otimes T_N + T_N \otimes I$ *is*

$$I \otimes T_N + T_N \otimes I = (Z \otimes Z) \cdot (I \otimes \Lambda + \Lambda \otimes I) \cdot (Z \otimes Z)^T. \tag{6.18}$$

$I \otimes \Lambda + \Lambda \otimes I$ *is a diagonal matrix whose* $(iN + j)$*th diagonal entry, the* (i, j)*th eigenvalue of* $T_{N\times N}$, *is* $\lambda_{i,j} = \lambda_i + \lambda_j$. $Z \otimes Z$ *is an orthogonal matrix whose* $(iN + j)$*th column, the corresponding eigenvector, is* $z_i \otimes z_j$.

Proof. From parts 1 and 3 of Lemma 6.3, it is easy to verify that $Z \otimes Z$ is orthogonal, since $(Z \otimes Z)(Z \otimes Z)^T = (Z \otimes Z)(Z^T \otimes Z^T) = (Z \cdot Z^T) \otimes (Z \cdot Z^T) = I \otimes I = I$. We can now verify equation (6.18):

$$
\begin{aligned}
&(Z \otimes Z) \cdot (I \otimes \Lambda + \Lambda \otimes I) \cdot (Z \otimes Z)^T \\
&\quad = (Z \otimes Z) \cdot (I \otimes \Lambda + \Lambda \otimes I) \cdot (Z^T \otimes Z^T) \\
&\qquad\qquad \text{by part 3 of Lemma 6.3} \\
&\quad = (Z \cdot I \cdot Z^T) \otimes (Z \cdot \Lambda \cdot Z^T) + (Z \cdot \Lambda \cdot Z^T) \otimes (Z \cdot I \cdot Z^T) \\
&\qquad\qquad \text{by part 1 of Lemma 6.3} \\
&\quad = (I) \otimes (T_N) + (T_N) \otimes (I) \\
&\quad = T_{N\times N}.
\end{aligned}
$$

Also, it is easy to verify that $I \otimes \Lambda + \Lambda \otimes I$ is diagonal, with diagonal entry $(iN + j)$ given by $\lambda_j + \lambda_i$, so that equation (6.18) really is the eigendecomposition of $T_{N\times N}$. Finally, from the definition of Kronecker product, one can see that column $iN + j$ of $Z \otimes Z$ is $z_i \otimes z_j$. □

The reader can confirm that the eigenvector $z_i \otimes z_j = \mathrm{vec}(z_j z_i^T)$, thus matching the expression for an eigenvector in equation (6.12).

For a generalization of Proposition 6.1 to the matrix $A \otimes I + B^T \otimes I$, which arises when solving the Sylvester equation $AX - XB = C$, see Question 6.5 (and Question 4.6).

Similarly, Poisson's equation in three dimensions leads to

$$T_{N\times N\times N} \equiv T_N \otimes I_N \otimes I_N + I_N \otimes T_N \otimes I_N + I_N \otimes I_N \otimes T_N,$$

with eigenvalues all possible triple sums of eigenvalues of T_N, and eigenvector matrix $Z \otimes Z \otimes Z$. Poisson's equation in higher dimensions is represented analogously.