2007-02-13 Differential Equations II, 2D1255, Homework problem 2

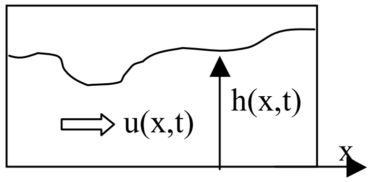
Hand in report no later than March 8.

Linear and nonlinear hyperbolic systems

In this exercise we shall investigate the relation between a non-linear problem and the corresponding linearized system. In particular we will see how well linear analysis predicts the behavior of the nonlinear problem.

Consider an initial-boundary value problem for the shallow water equations, L Ch 13.5 p 254:

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = 0, t \geq 0, 0 \leq x \leq 4,$$



$$\text{Initial conditions}: h(x,0) = 4 + \varepsilon e^{-\frac{(x-2)^2}{\delta^2}}, u(x,0) = 0$$

$$\text{Boundary conditions}: u(0,t) = u(4,t) = 0$$

h is the water height (m), u the velocity (m/s) and g the gravitational acceleration (10 m/s^2)

1. Numerical Solution

To begin with, let $\varepsilon = 0.1$ and solve numerically using the Lax-Friedrichs method. Use ghost cells at the boundaries. Prescribe values there by the procedure described in L Ch 7.3.3 for solid walls.

Note

a) 7.3.3 describes the linear acoustics (p,u) equations. Make an inspired guess that p corresponds to h.

b) The LF methods needs only ONE ghost cell (L decribes procedures with TWO ghost cells, necessary for the high-resolution methods).

c) The width of the pulse is δ. For a selected Δx, say $\approx$ 0.04, try different δ from broad to thin compared to Δx.

Choose Δx and Δt after making numerical experiments with the discretization parameters, Δx in the range 0.01 to 0.2. Take $\Delta t/\Delta x$ as large as possible, without violating stability. Make plots showing wave propagation and reflections at the boundaries. Compute at least until waves have been reflected at both boundaries and crossed each other. Run the program again with larger values of ε (= 1, 2, ..). How does the solution change?

2. Linearization

Choose a relevant constant state and derive the linearized problem at that state. Don't forget initial and boundary conditions. Show that the linear problem is hyperbolic by computing the characteristics, i.e. the wave speeds. What is the CFL limit on the time-step? Compare to the experiments in 1.

3. Analysis of Linear Problem

The linear problem can be solved analytically. Determine the solution of the linear problem at for instance $t = 0.5$. Discuss how information propagates, if the boundary conditions cause reflections and when reflected waves will appear. Compare with the numerical results for the non-linear case.

4. Non-Reflecting Boundary Conditions

Derive boundary conditions for the linear problem that do not cause reflections. Formulate the corresponding conditions for the non-linear case. Implement the conditions in your program simply extrapolating all variables at the boundary, L Ch 7.2.1. How well does the method work?

5. Characteristics, non-linear system

Now try a dam break problem, L pp 259, with suitable δ,

$$h(x,0) = 4 + \frac{2}{\pi}\arctan(\frac{x-2}{\delta})$$

Plot the characteristics starting at $t = 0$, $x = jh$, $j = 0, 1, \ldots$, $h = 0.1$ (40 x 2 curves), like L p. 261. Can you identify the *expansion wave* and the *shock*?