Re: Lab 3, $D(\phi)$, Conservation, etc.

1. The model considers ϕ, the volume fraction solids, so $0 < \phi < 1$ is assumed.
This is guaranteed by the conservation law (1B),

$$\phi_t + (\phi^1(1-\phi)^b)_x = 0$$

which gives total variation, TV(ϕ), = const., but NOT by (1A).
For (1A),

$$\frac{D\phi}{Dt} = \phi_t + u(x)\phi_x = -\phi u_x \neq 0$$

so the total variation is not constant. You will see $\phi > 1$ at the shock between clear fluid $\phi = 0$ and $\phi = 1$ at the bottom, and that is what the equation gives but not what a complete model should do. It happens because the model is simplified, and the interparticle forces have been left out: they become large when ϕ grows close to 1 (close packing) to prevent $\phi > 1$.
Now, D should always be non-negative, so you should for instance take

$$D(\min(1, \max(0, \phi)))$$

to give a reasonable fix to the problem.

2. For the Riemann solver, the recipe on L. p xxx can be used. I would encourage you to convince yourself that this is consistent with the characteristics.
The expansion wave comes out like this: Let the discontinuity be at $x = 0$. Then the similarity solution $q(x,t) = C(x/t)$, $x/t = \xi$ to the Riemann problem for the conservation law $q_t + (f(q))_x = 0$ is

$$C'(\xi)\cdot\frac{-x}{t^2} + f'(C)\cdot C'(\xi)\cdot\frac{1}{t} = 0 \Rightarrow \begin{cases} C' = 0 \\ \xi = f'(C(\xi)) \end{cases}$$

Now, the solution $C' = 0$ gives $Q^{\text{Riemann}} = Q_{\text{left}}$ or Q_{right}. Q^{Riemann} in the transonic expansion wave is given by $C(0) = Q^{\text{Riemann}}$, $0 = f'(C(0))$. i.e., the minimizer, if $f'' > 0$.

3. Conservation in 1A.
The flux function is $u\phi$ to be evaluated at $x_{i+1/2}$. *With the placement of gridpoints below the proper boundary condition for u is* $1/2(u_1+u_0) = 0$

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        Wall
  u0    I    u1         u2         u3   ...
  Q0    I    Q1         Q2         Q3   ...
  o----x----o----x----o----x----o----x----o—...
        I
     x=0     .5h  1h    1.5h 2h    2.5h ...
       x0         x1         x2         x3 ...
      F1/2       F3/2       F5/2
```

The flux at $x = 0$ is 0, and $f = u\phi$ satisfies this, of course. The Lax-Friedrichs flux function is not just f, but (see L p xxx)

$$F_{i-1/2} = \frac{1}{2}(f(Q_i) + f(Q_{i-1})) + \frac{\Delta x}{\Delta t}(Q_i - Q_{i-1})$$

so $F_{1/2} = \frac{1}{2}(u_1\phi_1 + \underbrace{u_0}_{-u_1}\phi_0) + \frac{\Delta x}{\Delta t}(\phi_1 - \phi_0)$ and the "reflection" $\phi_0 := \phi_1$ works. But if the u-gridpoints are positioned differently, another formula for the ghost point value must be used.