A function $g(x)$ on K is mapped as

$$g(x) = g(x^1)\,\Phi_1(X) + g(x^2)\,\Phi_2(X) + g(x^3)\,\Phi_3(X)$$

$$\nabla_x g(x) = g(x^1)\,\nabla_X \Phi_1 + g(x^2)\,\nabla_X \Phi_2 + g(x^3)\,\nabla_X \Phi_3$$

For integration: $dx = |J|\,dX$

Element stiffness matrix: $A^K_{ij} = \int_K \nabla\lambda_j \cdot \nabla\lambda_i \, dx$

$$A^K_{ij} = \int_{\hat{K}} \left(J^{-1} \nabla_X \Phi_j(X)\right) \cdot \left(J^{-1} \nabla_X \Phi_i(X)\right) \cdot |J| \, dX$$

Midpoint quadrature $\Rightarrow$

$$A^K_{ij} \approx \sum_{k=1}^{3} \left(J^{-1} \nabla_X \Phi_j(y^k)\right) \cdot \left(J^{-1} \nabla_X \Phi_i(y^k)\right) |J| \, \frac{|\hat{K}|}{3}$$

with $y^1 = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ $y^2 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ $y^3 = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$

and $|\hat{K}| = \frac{1}{2}$

Further reading:

CDE 15.1 : 2D Poisson eqn.
CDE 13 : Background
CDE 14.1–14.2 : 2D Mesh & p.w. polynomials
CDE 5.5, 14.4 : Quadrature
CDE 7 : optional (not part of the course)