Error estimation

Variational formulation: Find $u \in V$ s.t.

(V) $\int_0^1 a u' v' \, dx = \int_0^1 f v \, dx \quad \forall v \in V$

$V = \{ v : \|v\|^2 + \|v'\|^2 < \infty, \; v(0) = v(1) = 0 \}$

$a_1 \leq a(x) \leq a_2 \quad$ for $\quad 0 \leq x \leq 1$

Galerkin FEM: Find $U \in V_h$ s.t.

(G) $\int_0^1 a U' v' \, dx = \int_0^1 f v \, dx \quad \forall v \in V_h$

$V_h = \{ v : $ continuous & p.w. linear, ~~cont~~ on subintervals I_i, $v(0) = v(1) = 0 \}$

$V_h \subset V \Rightarrow$ (V) $-$ (G) : $\int_0^1 a (u-U)' v' \, dx = 0 \quad \forall v \in V_h$

"Galerkin orthogonality"

Weighted L_2-norm : $\|w\|_a = \left(\int_0^1 a w^2 \, dx \right)^{1/2}$

Energy norm : $\|v\|_E = \|v'\|_a = \left(\int_0^1 a (v')^2 \, dx \right)^{1/2}$

A priori error estimate:

$$\begin{aligned}
\|(u-U)'\|_a^2 &= \int_0^1 a (u-U)' (u-U)' \, dx \\
&= \int_0^1 a (u-U)' (u-v)' \, dx + \int_0^1 a (u-U)' (v-U)' \, dx \\
&= \int_0^1 a (u-U)' (u-v)' \, dx \leq \|(u-U)'\|_a \|(u-v)'\|_a
\end{aligned}$$

G.O.

$\Rightarrow \|(u-U)'\|_a \leq \|(u-v)'\|_a \quad \forall v \in V_h$

FEM solution $U \in V_h$ optimal in energy norm!