A posteriori error estimate of L_2 error

$$\|e\|^2=\int_0^1 e(-(a\varphi')'+c\varphi)\,dx=\int_0^1(ae'\varphi'+ce\varphi)\,dx$$

$$=\int_0^1(au'\varphi'+cu\varphi)\,dx-\int_0^1(aU'\varphi'+cU\varphi)\,dx$$

$$=\int_0^1 f\varphi\,dx-\int_0^1(aU'\varphi'+cU\varphi)\,dx$$

$$=\int_0^1 f(\varphi-\pi_h\varphi)-\sum_{j=1}^{M+1}\int_{I_j}(aU'(\varphi-\pi_h\varphi)'+cU(\varphi-\pi_h\varphi))\,dx$$

Integrate by parts, using that boundary terms vanish since $(\varphi-\pi_h\varphi)(x_j)=0$ for all nodes x_j.

$$\|e\|^2=\int_0^1(f+(aU')'-cU)(\varphi-\pi_h\varphi)\,dx$$

$$=\int_0^1 h^2R(U)\,h^{-2}(\varphi-\pi_h\varphi)\,dx$$

$$\le\|h^2R(U)\|\,\|h^{-2}(\varphi-\pi_h\varphi)\|$$

with residual $R(U)=f+(aU')'-cU$

Interpolation error estimate $\|h^{-2}(\varphi-\pi_h\varphi)\|\le C_i\|\varphi''\|$

$$\Rightarrow\ \|e\|^2\le\|h^2R(U)\|\,C_i\|\varphi''\|\,\frac{\|e\|}{\|e\|}$$

Thm. 15.4

$$\|e\|\le C_i\|h^2R(U)\|\,\frac{\|\varphi''\|}{\|e\|}\le\boxed{S\,C_i\,\|h^2R(U)\|}$$

with stability factor $S=\max_{\xi\in L_2(I)}\dfrac{\|\varphi''\|}{\|\xi\|}$

where φ is the dual solution with e replaced by ξ as data.