The abstract problem is: Find $u \in V$ s.t. ②

(*) $\quad a(u,v) = L(v)$ for all $v \in V$.

Problem: Do such solutions $u \in V$ exist?

Problem: If so: is such a solution unique?

"Existence" & "Uniqueness"

Depending on $a(\cdot,\cdot)$, $L(\cdot)$, and V, we may prove existence & uniqueness of solutions.

- Assume that $a(\cdot,\cdot)$ is V-elliptic or coercive:

  There exist constant $\varkappa_1 > 0$: $a(v,v) \geq \varkappa_1 \|v\|_V^2 \quad \forall v \in V$

- $a(\cdot,\cdot)$ is continuous: there exist constant $\varkappa_2$ such that $|a(v,w)| \leq \varkappa_2 \|v\|_V \|w\|_V \quad \forall v,v \in V$

- $L(\cdot)$ is continuous: there exist constant $\varkappa_3$ such that $|L(v)| \leq \varkappa_3 \|v\|_V \quad \forall v \in V$

Lax-Milgram Theorem (Th 21.1)

Suppose $a(\cdot,\cdot)$ is a continuous, V-elliptic bilinear form on V, and $L(\cdot)$ is a linear form on V. Then there exist unique $u \in V$ satisfying (*), and $\|u\|_V \leq \frac{\varkappa_3}{\varkappa_1}$