(2.) Mult. by $-t\Delta u$ & integrate

$$\Rightarrow (\dot{u}, -t\Delta u) - (\Delta u, -t\Delta u) = 0$$

$$\left(\frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) = \frac{1}{2}\|\nabla u\|^2 + t(\nabla\dot{u}, \nabla u) = \frac{1}{2}\|\nabla u\|^2 + (\dot{u}, -t\Delta u)\right)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) - \frac{1}{2}\|\nabla u\|^2 + t\|\Delta u\|^2 = 0$$

Integrate in time from 0 to T $\Rightarrow$

$$\frac{T}{2}\|\nabla u(T)\|^2 + \int_0^T t\|\Delta u\|^2 dt = \frac{1}{2}\int_0^T \|\nabla u\|^2 dt \leq \frac{1}{4}\|u_0\|^2 \quad (1)$$

$$\Rightarrow \boxed{\frac{T}{2}\|\nabla u(T)\|^2 + \int_0^T t\|\Delta u\|^2 dt \leq \frac{1}{4}\|u_0\|^2} \quad (2)$$

(3.) Mult. by $t^2\Delta^2 u$ & integrate $\Rightarrow$

$$(\dot{u}, t^2\Delta^2 u) - (\Delta u, t^2\Delta^2 u) = 0$$

$$\left(\frac{1}{2}\frac{d}{dt}(t^2\|\Delta u\|^2) = t\|\Delta u\|^2 + t^2(\Delta\dot{u}, \Delta u) = t\|\Delta u\|^2 + (\dot{u}, t^2\Delta^2 u)\right)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(t^2\|\Delta u\|^2\right) - t\|\Delta u\|^2 - (\Delta u, t^2\Delta^2 u) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(t^2\|\Delta u\|^2\right) \leq t\|\Delta u\|^2 \qquad \left((\Delta u, t^2\Delta^2 u) = t^2\|\nabla(\Delta u)\|^2 \geq 0\right)$$

$$\Rightarrow \frac{T^2}{2}\|\Delta u(T)\|^2 \leq \int_0^T t\|\Delta u\|^2 dt \leq \frac{1}{4}\|u_0\|^2$$

$$\Rightarrow \boxed{\|\Delta u(T)\| \leq \frac{1}{\sqrt{2}\,T}\|u_0\|} \quad (3)$$