

Time discretization with the θ -method

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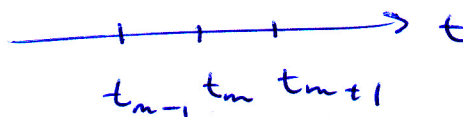
Mult. Heat eqn. with test function v

& integrate in space over Ω :

$$(u, v) + a(u, v) = 0 \quad \text{with } a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

Discretize in time

using the θ -method:



Find u_h^{m+1} ~~in~~ in V_h s. t. (for each time step)

$$\left(\frac{u_h^{m+1} - u_h^m}{k}, v \right) + a(u_h^{m+\theta}, v) = 0 \quad \forall v \in V_h$$

- $\theta = 0 \Rightarrow$ Forward Euler method in time
- $\theta = 1 \Rightarrow$ Backward Euler
- $\theta = 1/2 \Rightarrow$ Crank-Nicolson

Investigate the stability of the θ -method:

$$\text{Set } v = u_h^{m+\theta} \Rightarrow \left(\frac{u_h^{m+1} - u_h^m}{k}, u_h^{m+\theta} \right) + a(u_h^{m+\theta}, u_h^{m+\theta}) = 0$$

$$u_h^{m+\theta} = \theta u_h^{m+1} + (1-\theta) u_h^m = k \left(\theta - \frac{1}{2} \right) \frac{u_h^{m+1} - u_h^m}{k} + \frac{u_h^{m+1} + u_h^m}{2}$$

$$\Rightarrow k \left(\theta - \frac{1}{2} \right) \left\| \frac{u_h^{m+1} - u_h^m}{k} \right\|^2 + \frac{\|u_h^{m+1}\|^2 - \|u_h^m\|^2}{2k} + \|\nabla u_h^{m+\theta}\|^2 = 0 \quad (*)$$