Time discretization with the θ-method

Mult. Heat eqn. with test function v

& integrate in space over Ω:

$$(\dot{u}, v) + a(u, v) = 0 \qquad \text{with } a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

Discretize in time using the θ-method:

t_{m-1}, t_m, t_{m+1} → t

Find u_h^{m+1} ~~in~~ in V_h s.t. (for each time step)

$$\left(\frac{u_h^{m+1} - u_h^m}{k}, v\right) + a(u_h^{m+\theta}, v) = 0 \qquad \forall v \in V_h$$

$$\begin{cases} \theta = 0 \Rightarrow \text{Forward Euler method in time} \\ \theta = 1 \Rightarrow \text{Backward Euler} \text{ —— " ——} \\ \theta = 1/2 \Rightarrow \text{Crank-Nicolson} \end{cases}$$

Investigate the stability of the θ-method:

Set $v = u_h^{m+\theta} \Rightarrow \left(\frac{u_h^{m+1} - u_h^m}{k}, u_h^{m+\theta}\right) + a(u_h^{m+\theta}, u_h^{m+\theta}) = 0$

$$u_h^{m+\theta} = \theta u_h^{m+1} + (1-\theta) u_h^m = k\left(\theta - \frac{1}{2}\right) \frac{u_h^{m+1} - u_h^m}{k} + \frac{u_h^{m+1} + u_h^m}{2}$$

$$\Rightarrow k\left(\theta - \frac{1}{2}\right) \left\| \frac{u_h^{m+1} - u_h^m}{k} \right\|^2 + \frac{\|u_h^{m+1}\|^2 - \|u_h^m\|^2}{2k} + \|\nabla u_h^{m+\theta}\|^2 = 0 \quad (*)$$