Assume $\theta \in [1/2, 1] \Rightarrow \theta - 1/2 \geq 0$

$$\Rightarrow \frac{\|u_h^{m+1}\|^2 - \|u_h^m\|^2}{2k} + \|\nabla u_h^{m+\theta}\|^2 \leq 0$$

$$\Rightarrow \boxed{\|u_h^{n+1}\|^2 \leq \|u_h^n\|^2} \qquad (k \leq M)$$

$\Rightarrow$ θ-method unconditionally stable for $\theta \in [1/2, 1]$ (including backward Euler, C-N)

Forward Euler ($\theta = 0$) only conditionally stable:

stable for $k \leq C h^2$

$\Rightarrow$ very small time steps (stiff problem)

$\Rightarrow$ choose implicit method ($\theta > 0$).

"Proof": ($\ast$) $\Rightarrow$

$$\|u_h^{m+1}\|^2 - \|u_h^m\|^2 = \underbrace{-\left(2k\|\nabla u_h^{m+\theta}\|^2 + 2k^2\left(\theta - \tfrac{1}{2}\right)\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2\right)}_{D}$$

$D \leq 0 \Rightarrow$ stability: $\|u_h^{m+1}\|^2 \leq \|u_h^m\|^2$

$$D \leq 0 \Rightarrow -\left(2k\|\nabla u_h^{m+\theta}\|^2 + 2k^2\left(\theta - \tfrac{1}{2}\right)\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2\right) \leq 0$$

$$\Rightarrow 2k\|\nabla u_h^{m+\theta}\|^2 + 2k^2\left(\theta - \tfrac{1}{2}\right)\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2 \geq 0$$

$$\theta = 0 \Rightarrow \cancel{2}\cancel{k}\|\nabla u_h^m\|^2 \geq k^{\cancel{2}}\left\|\frac{u_h^{m+1} - u_h^m}{k}\right\|^2 \sim k\|\Delta u_h^m\|^2$$

$$\nabla u_h^m \sim \frac{u_h^m}{h}, \ \Delta u_h^m \sim \frac{u_h^m}{h^2} \Rightarrow \left(\frac{u_h^m}{h}\right)^2 \gtrsim k\left(\frac{u_h^m}{h^2}\right)^2 \Rightarrow \boxed{k \lesssim h^2}$$