$$U(x)=\sum_{j=1}^{M} \xi_j \phi_j(x) \rightarrow A\xi = b \qquad A=(a_{ij}),\ b=(b_i)$$

$$\begin{cases} a_{ij} = \int_0^1 \varepsilon \phi_j'(x)\phi_i'(x)\,dx + \int_0^1 \phi_j'(x)\phi_i(x)\,dx \\ b_i = -\int_0^1 \varepsilon u(0)\phi_0'(x)\phi_i'(x)\,dx + \int_0^1 u(0)\phi_0'(x)\phi_i(x)\,dx = -\int_0^1 (\varepsilon\phi_0'\phi_i' + \phi_0'\phi_i)\,dx \end{cases}$$

$$a_{ii} = \int_{I_i}\left(\varepsilon \frac{1}{h}\frac{1}{h} + \frac{1}{h}\frac{x-x_{i-1}}{h}\right)dx + \int_{I_{i+1}}\left(\varepsilon\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) + \left(-\frac{1}{h}\right)\frac{x_i - x}{h}\right)dx = \frac{\varepsilon}{h}+\frac{1}{2}+\frac{\varepsilon}{h}-\frac{1}{2} = \frac{2\varepsilon}{h}$$

$$a_{i-1\,i} = \int_{I_i}\left(\varepsilon\left(-\frac{1}{h}\right)\frac{1}{h} + \left(-\frac{1}{h}\right)\frac{x-x_{i-1}}{h}\right)dx = -\frac{\varepsilon}{h}-\frac{1}{2}$$

$$a_{i\,i+1} = \int_{I_{i+1}}\left(\varepsilon\frac{1}{h}\left(-\frac{1}{h}\right) + \frac{1}{h}\frac{x_i-x}{h}\right)dx = -\frac{\varepsilon}{h}+\frac{1}{2}$$

A non-symmetric!

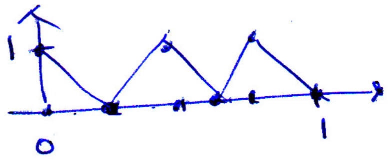
Equation i: $\sum_{j=1}^{n} \xi_j a_{ij} = \xi_{i-1}\left(-\frac{\varepsilon}{h}-\frac{1}{2}\right) + \xi_i \frac{2\varepsilon}{h} + \xi_{i+1}\left(-\frac{\varepsilon}{h}+\frac{1}{2}\right) = 0$ (for $i>1$).

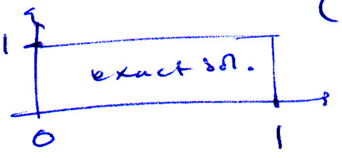
$\left(\frac{\varepsilon}{h}\right)$ large $\rightarrow -\xi_{i-1} + 2\xi_i - \xi_{i+1} = 0$ (Poisson-like...)

$\left(\frac{\varepsilon}{h}\right)$ small $\rightarrow -\frac{1}{2}\xi_{i-1} + \frac{1}{2}\xi_{i+1} = 0 \Leftrightarrow \xi_{i+1} = \xi_{i-1}$

$\Rightarrow$ solution ξ oscillating (M even) or A singular (M odd)

M even: $\varepsilon = 0$:

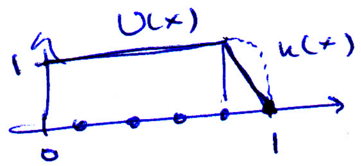




Stabilization: add artificial viscosity: $\varepsilon = \frac{h}{2}$

$\Rightarrow$ New equations: $\boxed{-\xi_{i-1} + \xi_i = 0}$ (upwind method)

$\Rightarrow$



- Galerkin FEM optimal for diffusion dominated problems
- ——— " ——— not optimal for convection ——— " ———
- For non-smooth exact solutions U contains spurious oscillations when using standard FEM when $\frac{\varepsilon}{h}$ small.
- Artificial viscosity $\varepsilon \sim h \rightarrow$ stability (no oscillations) but bad accuracy (no resolution of layers).