Navier–Stokes equations

$$\begin{cases} \dot{u} + u\cdot\nabla u + \nabla p - \nu\Delta u = f & \text{in } \Omega\times(0,T] \\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,T] \\ u = 0 & \text{on } \Gamma\times(0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

V.F. Find $(u,p)\in V\times Q$:

$$(\dot{u},v) + (u\cdot\nabla u, v) - (p,\nabla\cdot v) + (\nu\nabla u,\nabla v) + (\nabla\cdot u, q) = (f,v) \quad \text{for all } (v,q)\in V\times Q$$

FEM: Find $(U,P)\in V_h\times Q_h$:

$$(\dot{U},v) + (U\cdot\nabla U, v) - (P,\nabla\cdot v) + (\nu\nabla U,\nabla v) + (\nabla\cdot U, q) = (f,v) \quad \forall (v,q)\in V_h\times Q_h$$

GLS: Find $(U,P)\in V_h\times Q_h$:

$$(\dot{U},v) + (U\cdot\nabla U, v) - (P,\nabla\cdot v) + (\nu\nabla U,\nabla v) + (\nabla\cdot U, q) + (\delta(U\cdot\nabla U+\nabla P), U\cdot\nabla v + \nabla q) = (f, v+\delta(U\cdot\nabla v+\nabla q))$$

for all $(v,q)\in V_h\times Q_h$

Set $v=u$, $q=p$ in (V.F.) using that $(u\cdot\nabla u, u)=0$ (with $f=0$)

$$(\dot{u},u) + (\nu\nabla u,\nabla u) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\sqrt{\nu}\nabla u\|^2 = 0 \Rightarrow \|u(T)\|^2 + \int_0^T \|\sqrt{\nu}\nabla u\|^2 dt = \|u_0\|^2$$

Set $(v,q) = (U,P)$ in (GLS) $\Rightarrow$

$$\|U(T)\|^2 + \int_0^T \|\sqrt{\nu}\nabla U\|^2 dt + \int_0^T \|\sqrt{\delta}(U\cdot\nabla U+\nabla P)\|^2 dt = \|u_0\|^2$$
