# Courses/FEM/modules/functions

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## Contents



# Function Approximation

## Precondition

- Science

## Theory

### Function space

A vector space can be constructed with functions (a set of polynomials on a domain $\Omega$ for example) as vectors, where function addition and scalar multiplication satisfies the requirements for a vector space.

We can also define an inner product space with the L2 inner product defined as:

$$(f,g)_{L_2} = \int_\Omega f(x)g(x)dx$$

The inner product generates the norm:

$$\|f\|_{L_2} = \sqrt(f,f)$$

Just like in $R^d$ we define orthogonality between two vectors as:

$$(f,g)_{L_2} = 0$$

[Polynomial space]

[Piecewise polynomials]

### Polynomial interpolation

We can construct an interpolant $\pi f \in P^q(a,b)$ of the function $f$ by requiring that:

$$\pi f(\xi_i) = f(\xi_i), \quad i \in [0,...,q]$$
$$a \le \xi_0 < ... < \xi_q \le b$$

Assume that $f$ has $q+1$ continuous derivatives in $(a,b)$ and let $\pi f \in P^q(a,b)$ interpolate $f$ at the points $a \le \xi_0 < ... < \xi_q \le b$. Then for $a \le x \le b$:

$$|f(x) - \pi f(x)| \le |\frac{(x-\xi_0)...(x-\xi_q)}{(q+1)!}| max_{[a,b]} |D^{q+1} f|$$

### L2 (orthogonal) projection

An L2 projection $Pf$ is a projection of a function $f$ in the function space A to the function space B. We can think of this problem as solving the equation:

$$R(Pf) = Pf - f = 0, \quad x \in \Omega$$

However, since Pf and f belong to different function spaces, the residual $R(Pf)$ can in general not be zero. The best we can hope for is that $R(Pf)$ is orthogonal to B, which means solving the equation:

$$(Pf - f, v) = 0, \quad x \in \Omega, \quad \forall v \in B$$

#### The $L_2$ projection is the best possible approximation

The orthogonality condition means that $Pf$ is the best possible approximation in B, i.e. we cannot pick an object $v \in B$ which is a better approximation that $Pf$ in the $L_2$ norm:

$$\|f - Pf\|^2 = (f - Pf, f - Pf) = (f - Pf, f - v) + (f - Pf, v - Pf) = [v - Pf \in B] = (f - Pf, f - v) \le \|f - Pf\| \|f - v\|$$
$$\Rightarrow$$
$$\|f - Pf\| \le \|f - v\|, \quad \forall v \in B$$

#### $L_2$ projection error estimate

Since $\pi f \in B$, we can choose $v = \pi f$ which gives:

$$\|f - Pf\| \le \|f - \pi f\|$$

i.e. we can use an interpolation error estimate since it bounds the projection error.

## Software

To compute the L2 projection we want to solve the equation: $(R(u), v) = (u, v) - (f, v) = 0$. We identify the terms with u and v as **bilinear forms**: $a(u,v)$ and the terms with only v as **linear forms**: $L(v)$.

We can thus define the representation of the equation in FEniCS as:

```
u = TrialFunction(element)
v = TestFunction(element)

a = (u * v) * dx
L = (f * v) * dx
```

Construct discrete functions and compute functionals. Compute an $L_2$ projection of a given function and compute the functional (norm) $\|f - Pf\|$.

A form without u and v is identified as $M$ (a functional), and can be used to compute a norm of a known function for example:

```
u = TrialFunction(element)
v = TestFunction(element)

M = (f * f) * dx
```

Assembly:

A form `q` is assembled by:

```
T = assemble(q, mesh)
```

This is the same basic algorithm for assembling a matrix (from a bilinear form), vector (from a linear form) or scalar (from a functional).

Vector indexing:

You can get and set values of a vector by a standard bracket notation:

```
a = x[3]
x[4] = a + 4
```

## Postcondition

You should now be familiar with:

- L2 inner product
- interpolation
- L2 projection
- Mesh size function $h$

## Exercises

- CDE: 5.7, 5.8, 5.14, 5.17, 5.1/5.21

#### Problem

We let $\mathcal{P}^q(a,b)$ denote the set of polynomials $p(x) = \sum_{i=0}^{q} c_i x^i$ of degree at most $q$ on an interval $(a,b)$, where the $c_i \in$ are called the coefficients of $p(x)$. We recall that two polynomials $p(x)$ and $r(x)$ may be added to give a polynomial $p+r$ defined by $(p+r)(x) = p(x) + r(x)$ and a polynomial $p(x)$ may be multiplied by a scalar $\alpha$ to give a polynomial $\alpha p$ defined by $(\alpha p)(x) = \alpha p(x)$. Similarly, $\mathcal{P}^q(a,b)$ satisfies all the requirements to be a vector space where each ``vector *is a* particular polynomial function $p(x)$.

Prove this claim.

#### Problem

Compute formulas for the linear interpolant of a continuous function f through the points a and (b + a)/2. Plot the corresponding Lagrange basis functions.

#### Problem

Write down the polynomial of degree 3 that interpolates $sin(x)$ at $\xi_0 = 0, \xi_1 = \pi/6, \xi_2 = \pi/4$, and $\xi_3 = \pi/3$, and plot $p_3$ and sin on $[0, \pi/2]$.

## Examination

1.1

Construct linear Lagrange basis functions with the nodes in the vertices on a tetrahedron in 3D.

1.2

Using FEniCS:

Compute the L2 projection Pf of a function f (a trigonometric function for example) in 1D or 2D on a space of piecewise linear polynomials with just a few points. Compute the L2 norm of the error $\|f - Pf\|$ as a functional in FEniCS. Try to choose better values of the coefficients $\xi_j$ in $Pf = \sum_j^M \xi_j \phi_j$. Are you able to reduce the error?

Note: Expressions with f will be integrated with quadrature (f is represented as a finite element function on each cell). Choose a higher order representation for f to remove the effect of the quadrature error on the result, i.e.:

```
element2 = FiniteElement("Lagrange", "triangle", 2)
f = Source(element2, mesh)
```

represents f as a quadratic rather than linear function when integrating.

## TODO

- Polynomial space
- Piecewise polynomials
- Software part