FEM08 - lecture 6

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Implementation of adaptivity

Recall error bound:
Split up error bound into contributions from each cell:

$$C\|h\hat{R}(U)\| = C\sqrt{\int_0^1 (h\hat{R}(U))^2 dx} = C\sqrt{\sum_K \int_K (h\hat{R}(U))^2 dx}$$

Simple condition: refine (split) cells K where the error contribution (error indicator) $\int_K (h\hat{R}(U))^2 dx$ is largest.

Implementation of adaptivity

How to compute array of error indicators:

```
DV = FunctionSpace(mesh, ''DG'', 0)
v = TestFunction(DV)
h = MeshSize(''triangle'', mesh)
Lei = (h*R)*(h*R)*v*dx
eix = assemble(Lei, mesh)
```

where DV is the space of piecewise constant functions (constant on each triangle, discontinuous)

Effect: $v = 1$ on each triangle, compute integral contribution from each triangle.

Implementation of adaptivity

How to refine the mesh:

```
cell_markers = MeshFunction(‘‘bool’’, mesh, mesh.topology().dim())
cell_markers.fill(False)

for c in cells(mesh):
    if(condition):
        cell_markers.set(c, True)

mesh.refine(cell_markers)
```

FEM for ODE

In general:
$\dot{u} = f(t, u)$

Model problem:
$\dot{u} + au = g(t)$

Recall Forward Euler method:
$U_n = U_{n-1} + kf(t_{n-1}, U_{n-1}) = U_{n-1} + k(-aU_{n-1} + g(t_{n-1}))$
other methods of similar form (Backward Euler, Crank-Nicolson)

FEM for ODE: cG(1)

cG(1): $U \in V_k$ with $U(0) = u_0$
V_k space of piecewise linears, W_k space of piecewise constants

$$\int_{t_{n-1}}^{t_n} \dot{U}v + aUv - gv dt = 0, \quad \forall v \in W_k \Rightarrow$$

$$U = \sum_{i=1}^{N} \xi_i \phi_i$$

$$\phi_n = \frac{t - t_{n-1}}{k_n}, \phi_{n-1} = \frac{t_n - t}{k_n}, k_n = t_n - t_{n-1} \Rightarrow$$

$$\xi_n = \xi_{n-1} - \int_{t_{n-1}}^{t_n} a(t)(\xi_{n-1}\phi_{n-1} + \xi_n\phi_n) + g(t_n) dt$$

FEM for ODE: cG(1)

Trapezoid quadrature, we get Crank-Nicolson (+ quad. err.):

$$\xi_n = \xi_{n-1} - \frac{1}{2} k_n (a(t_{n-1})\xi_{n-1} + a(t_n)\xi_n + g(t_{n-1}) + g(t_n)) + E_q$$

Error estimate for cG(1)

For general ODE: $R(u) = \dot{u} - f(t,u) = 0$
Similar construction as for error estimates in space.

$$|u(T) - U(T)| \leq S(T)k|\hat{R}(U)|$$

where stability factor $S(T) = \frac{\int_0^T |\dot{\phi}| dt}{e_T}$

S gives a quantitative measure of the stability of the equation.

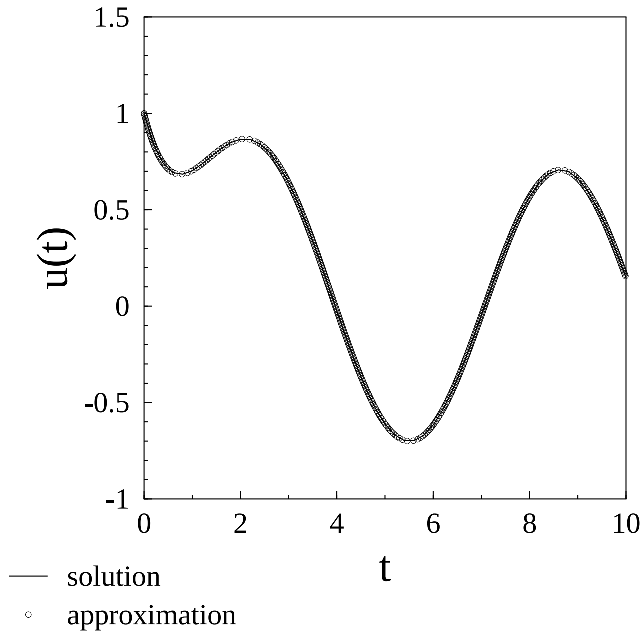
Stability factor examples

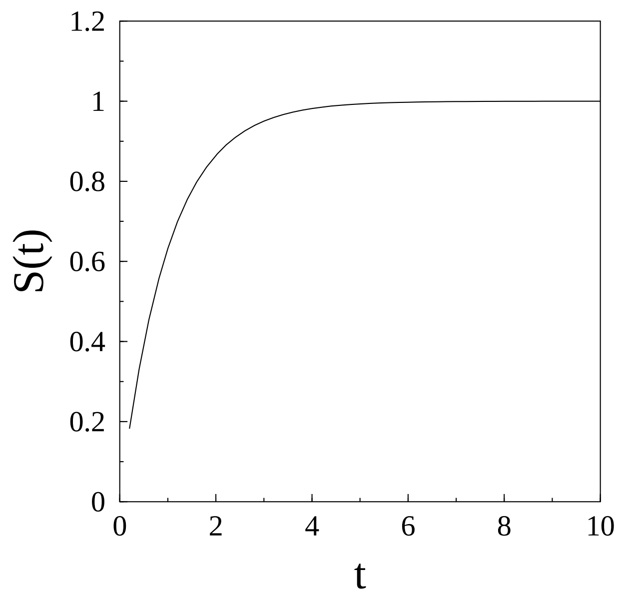
Primal equation

$\dot{u} + u = sin(t), \quad u(0) = u_0$

Dual equation

$-\dot{\phi} + u = 0, \quad \phi(T) = e_T$





S doesn't grow with $T \Rightarrow$ equation is very stable/*parabolic*.

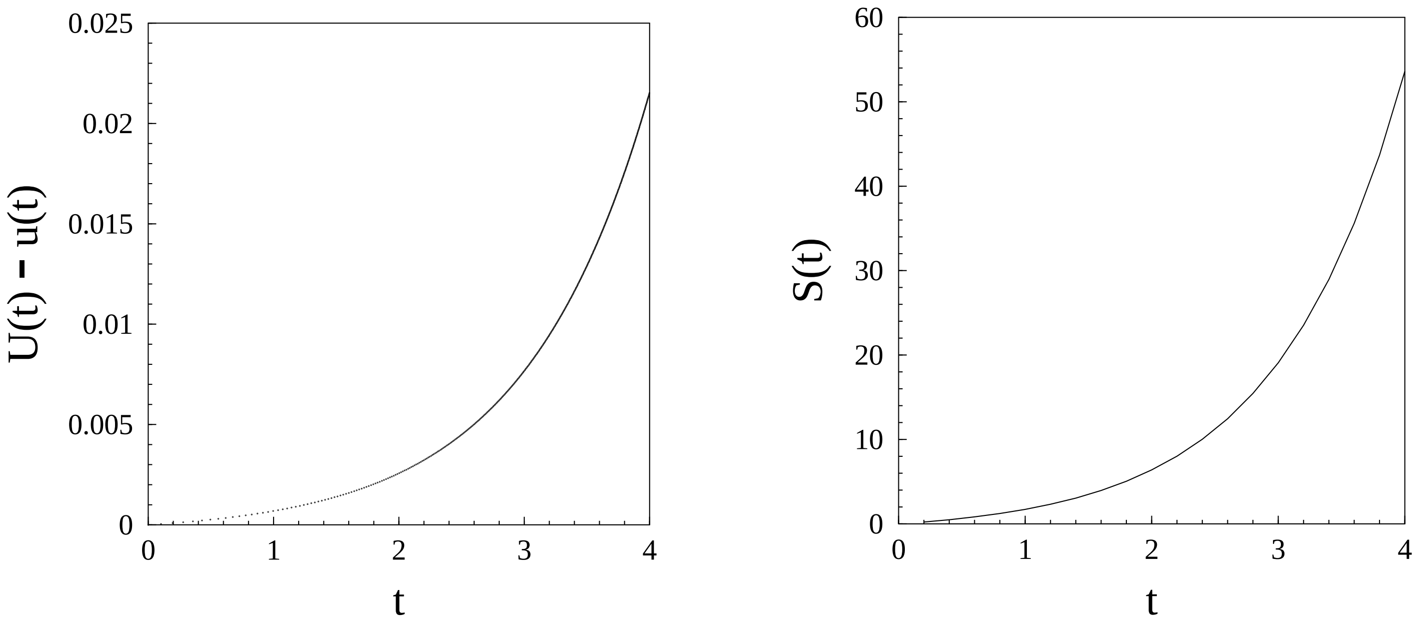
Stability factor examples

Primal equation

$\dot{u} - u = f(t), \quad u(0) = u_0$

Dual equation

$-\dot{\phi} - u = 0, \quad \phi(T) = e_T$



S grows exponentially with $T \Rightarrow$ very unstable/expensive to compute accurately.

FEM for IBVP/PDE (space-time)

Model problem (Heat equation):
$\dot{u} - \Delta u = f(t, x)$

Domain D is cartesian product of domain in space and time interval: $D = \Omega \times I$

Mesh is space-time slab $D_n = T_n \times I_n$ where $T_n = K$ is triangulation of Ω and I_n is sub-interval of length $k_n = t_n - t_{n-1}$.

cG(1)dG(0): Trial and test space $\bar{W}_k$ with basis functions $\bar{\phi} = \phi_t \phi_x$, where ϕ_t basis functions of W_k (piecewise constant/discontinuous in time) and ϕ_x basis functions of V_h (piecewise linear/continuous in space).

FEM for IBVP/PDE (space-time)

Galerkin's method for cG(1)dG(0):

$$(U_n, v) = (U_{n-1}, v) - \int_{t_{n-1}}^{t_n} (\nabla U, \nabla v)) + (f, v)dt, \quad \forall v \in \bar{W}_k$$

$$U = \sum_{i=1}^{N} \xi_i \bar{\phi} = \sum_{i=1}^{N} \xi_i \phi_{xi}, \quad v = \phi_{xi}, \quad i = 1, ..., N$$

Again, with right-endpoint quadrature we get backward Euler (+quad. err.):

$$(U_n, v) = (U_{n-1}, v) - k_n(\nabla U(t_n, x), \nabla v)) + (f(t_n), v), \quad \forall v \in \bar{W}_k$$

FEM for IBVP/PDE (space-time)

Substituting U gives the formulas for matrix/vector elements.

Appendix:

A posteriori/duality 2D/3D

We have *primal equation*:
$-\nabla \cdot (a\nabla u) - f = 0$
We introduce *dual equation*:
$-\nabla \cdot (a\nabla \phi) - \psi = 0$
which is defined by:
$(Av, w) = (v, A^* w)$
where A is the diff. op. for the primal eq. and A^* for the dual eq.
(in this case they are the same).
(Homogenous Dirichlet BC)

Duality 2D

We compute solution U by Galerkin's method:
$\int_\Omega -a\nabla U \cdot \nabla v - fvdx = 0, \quad \forall v \in V_h, \quad U \in V_h$

Error $e = u - U$ satisfies:
$\int_\Omega a\nabla e \cdot \nabla wdx = \int_\Omega a\nabla U \cdot \nabla w - fwdx$

Duality 2D

Want to bound quantity $M(e) = (e, \psi)$:

$$(e, \psi) = \int_\Omega e\psi dx = \int_\Omega e(-\nabla \cdot (a\nabla\phi))dx =$$

$$\int_\Omega a\nabla e\nabla\phi dx + \int_\Gamma e(\nabla\phi) \cdot n]ds = [e = 0, x \in \Gamma] =$$

$$\int_\Omega a\nabla U \cdot \nabla\phi - f\phi dx =$$

$$[GO] = \int_\Omega -a\nabla U \cdot \nabla(\phi - \pi\phi) + f(\phi - \pi\phi)dx =$$

$$\sum_{K_i} \int_{K_i} (\nabla(a\nabla U) + f)(\phi - \pi\phi)dx + \int_{\partial K_i} a(\nabla U) \cdot n(\phi - \pi\phi)ds$$

Duality 2D

We get facet (∂K_i) integrals from both cells sharing the facet F. We write the sum (normals are opposite, so they have opposite signs):

$$\int_F [a(\nabla U) \cdot n](\phi - \pi\phi) ds$$

Duality 2D

Two ways to continue: use interpolation estimate for boundary expression or express boundary integral as interior integral:

$$\int_F [a(\nabla U) \cdot n](\phi - \pi\phi)ds =$$

$$\int_F h^{-1}[a(\nabla U) \cdot n](\phi - \pi\phi)hds \approx$$

$$\int_F h^{-1}[a(\nabla U) \cdot n](\phi - \pi\phi)dx$$

We can then continue with an interior integral:

$$\sum_{K_i} \int_{K_i} (\nabla(a\nabla U) + f + h^{-1}[a(\nabla U) \cdot n)])(\phi - \pi\phi)dx$$

Duality 2D

Where we can now just like for 1D use Cauchy-Schwartz and an interpolation estimate:

$$|(e,\psi)| = \sum_{K_i} \int_{K_i} (\nabla(a\nabla U) + f + h^{-1}[a(\nabla U)\cdot n]])(\phi - \pi\phi)dx \leq$$

$$\|\hat{R}(U)\|\|\phi - \pi\phi\| \leq h^2 \|\hat{R}(U)\|\|D^2\phi\|$$

where we identify $S = \|D^2\phi\|$ as a *stability factor*.
and $\hat{R}(U) = \nabla(a\nabla U) + f + h^{-1}[a(\nabla U)\cdot n)]$ piecewise constant (constant on each cell).