DN2260 Finite Element Methods: Written Examination

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Answers should be given in English. Read all the questions before starting the work. Do the easy ones first. All answers should be motivated, answers with insufficient explanations may reduce the credits. A total of 20 out of max 50 points guarantees a pass (3, ECTS E).

P1. (10p)

Consider the problem:

$$\begin{cases} -\Delta u = f, & x \in \Omega \subset \mathbb{R}^2, \\ u(x) = g, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $g = g(x)$ and $f = f(x)$ are given functions.

(5p)(a) Assume 1D case of the problem (1):

$$\begin{cases} -u'' = 1, & x \in (0,3), \\ u(0) = 0, \\ u(3) = 1. \end{cases} \tag{2}$$

Derive a finite element method using continuous piecewise linear approximation and derive a discrete linear system for nodes: $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$. How many unknowns does the discrete linear system have? Write the system matrix and right-hand side vector.

(5p)(b) Formulate a finite element method using continuous piecewise linear approximation defined on the mesh in Fig. 1 for homogeneous Dirichlet boundary condition $g(x) = 0$ on $\partial\Omega$, $f(x) = 1$ in Ω. Compute the corresponding matrix and vector.

P2.(10p)

Consider the following differential equation:

$$\begin{cases} -u''(x) + cu(x) = f(x), & x \in (0,1), c \geq 0, \\ \quad u(0) = u(1) = 0. \end{cases} \tag{3}$$

(2p)(a) Derive a weak formulation and Galerkin discretization with appropriate functional spaces, V and V_h. Write the Galerkin orthogonality for the error $e = u - U$, where $u \in V$, $U \in V_h$.

Futher, we define the residual of (3) by $R(U) = -U'' + cU - f$.

(4)(b) Prove a priori and a posteriori error estimates in the energy norm $||v||_E$, $||v||_E^2 = ||v'||^2_{L_2(0,1)} + c||v||^2_{L_2(0.1)}$.

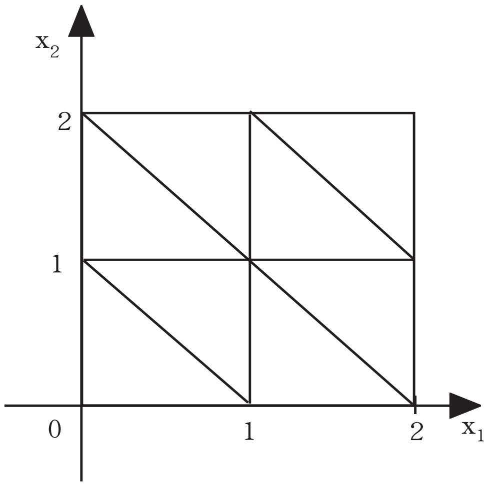

Figure 1: Triangulation of the domain Ω.

(4)(c) Prove the a posteriori error estimate $||u-U||^2_{L_2(0,1)} \leq SC_i||h^2R(U)||_{L_2(0,1)}$, where $S = ||\phi''||_{L_2(0,1)}$ is the stability factor, C_i is an interpolation constant, h is the mesh size, and ϕ solves the following corresponding dual problem:

$$\begin{cases} -\phi''(x) + c\phi(x) = e, & x \in (0,1), c \geq 0, \\ \phi(0) = \phi(1) = 0. \end{cases} \tag{4}$$

P3. (10p)

Consider the problem:

$$\begin{cases} -\Delta u(x) + \alpha u(x) = f(x), & x \in \Omega \subset \mathbb{R}^3, \\ \beta \partial_n u(x) + \gamma u(x) = g(x), & x \in \Gamma, \end{cases} \tag{5}$$

with $\partial_n u = \nabla u \cdot n$, n the outward normal of the boundary Γ, and α, β, γ are non-negative constants.

State the Lax-Milgram theorem. Determine if the assumptions of the Lax- Milgram theorem are satisfied in the following cases:

(3p)(a) $\alpha = 0, \beta = 0, \gamma = 1, g = 0, f \in L_2(\Omega)$
(4p)(b) $\alpha = 0, \beta = 1, \gamma = 0, g = 0, f \in L_2(\Omega)$
(3p)(c) $\alpha = 1, \beta = 1, \gamma = 0, g \in L_2(\Gamma), f = 0$

P4.(10p)

Consider the following initial-boundary value problem

$$\begin{cases} \dot{u} + \beta \cdot \nabla u - \varepsilon \Delta u = 0, & (x,t) \in \Omega \times I, \\ u = 0, & (x,t) \in \partial\Omega \times I, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{6}$$

where $\varepsilon > 0$ is the diffusion coefficient and β is a constant convection velocity vector.

(6)(a) Assume $\beta = 0$ and $\varepsilon = 1$. Show the stability estimates

$$\begin{aligned} ||u(t)||^2_{L_2(\Omega)} + \int_0^t ||\nabla u(s)||^2_{L_2(\Omega)} ds &\leq ||u(0)||^2_{L_2(\Omega)} \\ ||\nabla u(t)||^2_{L_2(\Omega)} + \int_0^t ||\Delta u(s)||^2_{L_2(\Omega)} ds &\leq ||\nabla u(0)||^2_{L_2(\Omega)} \end{aligned} \tag{7}$$

(2)(b) What happens to the standard Galerkin FEM when $\beta \neq 0$ and ε is too small? What should ε be compared to to decide if it is small?

(2)(c) Describe a Petrov-Galerkin finite element method which avoids the problem in **(2)(b)**.

P5(10p)

Answer the following questions related to standard FEM algorithms (it may be helpful to illustrate some of your answers with pictures):

(3p)(a) Describe how a mapping to a reference element is used to compute element integrals, in the case of triangular elements (you do not have to carry out any computations, just illustrate the idea and computation of the algorithm).

(4p)(b) Describe the steps in an adaptive FEM. Explain the terms "local mesh refinement" and "a posteriori error estimates" and how these steps are used in the algorithm.

(3p)(c) With respect to mesh refinement, describe a "red-green" algorithm. What is a "hanging" node? How is the condition of no hanging nodes guaranteed in the algorithm?