School of Computer Science and Communication, KTH

2D1260 Finite Element Methods: Written Examination

Saturday 2006-01-21, kl 8-13

Coordinator: Johan Hoffman

Aids: none. **Time:** 5 hours.

Answers may be given in English or in Swedish. All answers should be explained and calculations shown unless stated otherwise. A correct answer without explanation can be left without points. Do not leave integrals or systems of equations unsolved unless explicitly allowed. *Each of the 5 problems gives 10 p, resulting in a total of 50 p: 20 p for grade 3, 30 p for grade 4, and 40 p for grade 5.*

Problem 1: Consider the problem:

$$
\begin{aligned}
-\Delta u(x) &= 1, \quad x \in \Omega \subset \mathbb{R}^2, \\
u(x) &= 0, \qquad x \in \partial\Omega,
\end{aligned}
$$

with $x = (x_1, x_2)$ and Ω the square defined in Fig. 1 (see next page).

(a) Formulate a finite element method (FEM) using a continuous piecewise linear approximation (cG(1)) defined on the mesh in Fig. 1.
(b) Compute the corresponding matrix and vector. You do not have to solve the resulting system of equations.
(c) Compute the corresponding matrix and vector, with now the homogeneous Dirichlet boundary condition $u = 0$ replaced by the Neumann condition

$$\frac{\partial u}{\partial x_1} = 0,$$

for $x_1 = 2, 0 < x_2 < 2$ (with still homogeneous Dirichlet boundary conditions for the rest of the boundary). You do not have to solve the resulting system of equations.

Note: *The exam continues on the next page!*

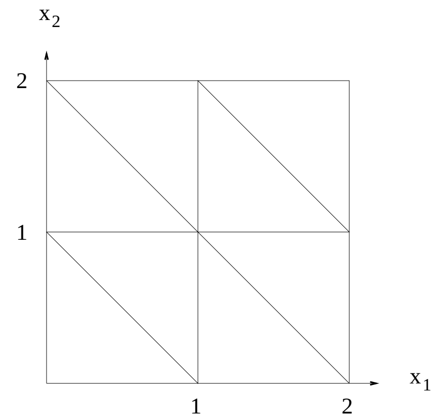

Figure 1: Triangulation (mesh) of domain Ω.

Problem 2: Consider the problem:

$$
\begin{aligned}
-\Delta u(x) + \alpha u(x) &= f(x), \quad x \in \Omega \subset \mathbb{R}^3, \\
\beta \partial_n u(x) + \gamma u(x) &= g(x), \qquad x \in \Gamma,
\end{aligned}
$$

with $\partial_n u = \nabla u \cdot n$, n the outward normal of the boundary Γ, and α,β, γ are non-negative constants.

State the Lax-Milgram theorem. Determine if the assumptions of the Lax-Milgram theorem are satisfied in the following cases:

(a) $\alpha = 0$, $\beta = 0$, $\gamma = 1$, $g = 0$, $f \in L_2(\Omega)$
(b) $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $g = 0$, $f \in L_2(\Omega)$
(c) $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $g \in L_2(\Gamma)$, $f = 0$

For each case (a)-(c); derive a bilinear form $a : V \times V \to \mathbb{R}$ and a linear form $L : V \to \mathbb{R}$, and specify the Hilbert space V and the norm $\| \cdot \|_V$.

Hint: The following Trace Inequality may be useful: There exist a constant C, such that for all $v \in H^1(\Omega)$, we have that

$$\|v\|_{L_2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}.$$

Note: *The exam continues on the next page!*

Problem 3: Consider an abstract variational problem: Find $u \in V$ such that

$$a(u, v) = L(v)$$

for all $v \in V$, with V a Hilbert space, and $a(\cdot, \cdot)$ and $L(\cdot)$ are bilinear and linear forms on V satisfying the conditions in Lax-Milgrams theorem. The abstract Galerkin method for this problem is formulated as: Find $U \in V_h$ such that

$$a(U, v) = L(v)$$

for all $v \in V_h$, with V_h a finite dimensional subspace of V.

(a) Prove the Galerkin orthogonality: $a(u - U, v) = 0$, for all $v \in V_h$.
(b) Prove that the Galerkin solution U is the best possible solution in the space V_h, with respect to the energy norm $\|w\|_E = \sqrt{a(w, w)}$, using the Schwarz inequality:

$$a(v, w) \leq \|v\|_E \; \|w\|_E, \quad \forall v, w \in V.$$

(c) Now consider the case of $V = H_0^1(0, 1)$, and $V_h = \{$continuous piecewise linear functions v on $\mathcal{T}_h$ with $v(0) = v(1) = 0\}$, with $\mathcal{T}_h$ a subdivision of the interval (0,1). Define

$$a(u, v) = \int_0^1 a(x) u'(x) v'(x) \; dx, \quad L(v) = \int_0^1 f(x) v(x) \; dx.$$

The energy norm $\| \cdot \|_E$ for this problem is defined as $\|v\|_E = \|v'\|_a$, with the weighted L_2 norm

$$\|w\|_a = \left(\int_0^1 a(x) w^2(x) \; dx \right)^{1/2}$$

Prove the a priori error estimate: $\|u - U\|_E \leq C_i \|hu''\|_a$
(d) Prove the a posteriori error estimate: $\|u - U\|_E \leq C_i \|hR(U)\|_{a^{-1}}$

The residual $R(U) = f + (aU')'$ is defined on each subinterval $I_i = (x_{i-1}, x_i)$, where x_i are the nodes, and C_i is an interpolation constant.

Note: *The exam continues on the next page!*

Problem 4: Consider the problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = 0, \quad \forall (x,t) \in \Omega \times (0,T], \\
&u(x,t) = 0, \quad \forall x \in \partial\Omega, \\
&u(x,0) = u_0(x).
\end{aligned}$$

Prove the following stability estimate:

$$\frac{1}{2}\|u(T)\|^2 + \int_0^T \|\nabla u\|^2 \, dt = \frac{1}{2}\|u_0\|^2,$$

where $\|\cdot\|$ is the standard L_2-norm of functions defined on Ω.

Problem 5: Answer the following questions related to standard FEM algorithms (it may be helpful to illustrate your answers with pictures):

(a) Why do we have to stabilize a convection dominated convection-diffusion equation on coarse meshes?
(b) What is artificial viscosity?
(c) What is a least squares stabilized finite element method?
(d) What is the empty circle property of a Delaunay triangulation?
(e) What is the motivation for including duality in a posteriori error estimation?

Good Luck!

Johan

Solutions to exam

Problem 1: See pages 360-363 in the CDE book.

(a) Find $U \in V_h$ such that

$$\int_\Omega \nabla U(x) \cdot \nabla v(x)\ dx = \int_\Omega v(x)\ dx \quad \forall v \in V_h \tag{1}$$

(b) $V_h = \{$continuous piecewise linear functions v on $\mathcal{T}_h$ such that v=0 on $\partial\Omega\}$, with $\mathcal{T}_h$ the triangulation of Ω in Fig. 2, with $h = 1$. There is 1 degree of freedom; the node N_1.

A basis for V_h is $\{\phi_1\}$; with $\phi_1 \in V_h$, and $\phi_1(N_1) = 1$ and $\phi_1 = 0$ in all other vertices (nodes). Set $U(x) = \xi_1\phi_1(x)$, then (1) is equivalent to $A\xi = b$ where A and b are scalars, given by

$$A_{11} = \int_\Omega \nabla\phi_1(x) \cdot \nabla\phi_1(x)\ dx, \quad b_1 = \int_\Omega \phi_1(x)\ dx$$

A_{11} involves integration over elements $e_2, e_3, e_4, e_5, e_6, e_7$, where e_2, e_7 are of the type in Fig.15.8 at page 362 in the CDE book, with integral $\int_{e_2} \nabla\phi_1 \cdot \nabla\phi_1\ dx = 1$, and e_3, e_4, e_5, e_6 are of the type in Fig.15.9, with integral $\int_{e_3} \nabla\phi_1 \cdot \nabla\phi_1\ dx = 1/2$. Thus

$$A_{11} = \int_{e_2} + \int_{e_3} + \int_{e_4} + \int_{e_5} + \int_{e_6} + \int_{e_7} = 1 + 1/2 + 1/2 + 1/2 + 1/2 + 1 = 4$$

$$b_1 = \int_\Omega \phi_1(x)\ dx = \text{volume under } \phi_1 = 6 \times \frac{\frac{h^2}{2} \times 1}{3} = h^2 = 1.$$

(c) Now we have 2 degrees of freedom: N_1, N_2, which leads to a 2×2-matrix A, and a 2-vector b, given by

$$A_{ij} = \int_\Omega \nabla\phi_i(x) \cdot \nabla\phi_j(x)\ dx, \quad b_i = \int_\Omega \phi_i(x)\ dx,$$

since the boundary integral from the partial integration is equal to zero by the boundary conditions. $A_{11} = 4$ as before. A_{22} involves integration over elements e_4, e_7, e_8, where e_4 is of the type in Fig.15.8 at page 362 in the CDE book, with integral $\int_{e_4} \nabla\phi_2 \cdot \nabla\phi_2\ dx = 1$, and e_7, e_8 are of the type in Fig.15.9, with integral $\int_{e_7} \nabla\phi_2 \cdot \nabla\phi_2\ dx = 1/2$. Thus

$$A_{22} = \int_{e_4} + \int_{e_7} + \int_{e_8} = 1 + 1/2 + 1/2 = 2$$

A_{12} involves integration over elements e_4, e_7, which are of the type in Fig.15.10 at page 363 in the CDE book, with integral $\int_{e_4} \nabla\phi_1 \cdot \nabla\phi_2 \, dx = -1/2$. Thus

$$A_{12} = \int_{e_4} + \int_{e_7} = -1/2 - 1/2 = -1,$$

and $A_{21} = A_{12}$.

$b_1 = 1$ as before, and

$$b_2 = \int_\Omega \phi_1(x) \, dx = \text{volume under } \phi_2 = 3 \times \frac{\frac{h^2}{2} \times 1}{3} = \frac{3}{6}h^2 = \frac{1}{2}.$$

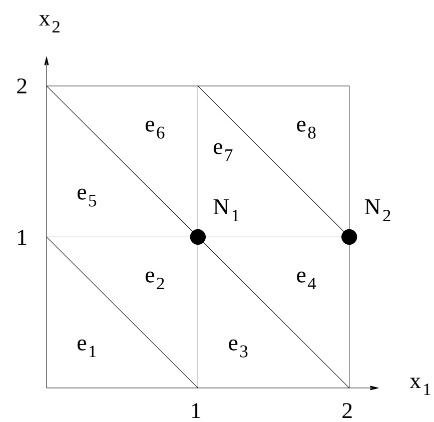

Figure 2: Triangulation (mesh) of domain Ω.

Problem 2: Theorem 21.1 in the CDE book.

(a) Section 21.4.3 in CDE book. Use the Poincare-Friedrich inequality (Theorem 21.4) to prove that $a(\cdot,\cdot)$ is elliptic. Continuity of $a(\cdot,\cdot)$ follows by

$$\begin{aligned} |a(v,w)| &= \int_\Omega \nabla v \cdot \nabla w \, dx \leq \|\nabla v\| \, \|\nabla w\| \\ &\leq (\|\nabla v\|^2 + \|v\|^2)^{1/2} (\|\nabla w\|^2 + \|w\|^2)^{1/2} = \|v\|_V \|w\|_V \end{aligned}$$

(b) The assumptions of the Lax-Milgram theorem are not satisfied: With $V = H^1(\Omega)$ we can prove that $a(\cdot,\cdot)$ and $L(\cdot)$ are continuous, but we cannot prove that $a(\cdot,\cdot)$ is elliptic; we cannot bound the L_2-norm of the solution using Poincare-Friedrich inequality (Theorem 21.4) since we do not know anything about the solution on the boundary (we have only Neumann boundary conditions).

(c) Section 21.4.4 in the CDE book. Continuity of $a(\cdot,\cdot)$ follows by

$$
\begin{aligned}
|a(v,w)| &= \int_\Omega (\nabla v \cdot \nabla w + vw)\; dx \leq \|\nabla v\|\; \|\nabla w\| + \|v\|\; \|w\| \\
&= (\|\nabla v\|, \|v\|) \cdot (\|\nabla w\|, \|w\|) \leq \|v\|_V \|w\|_V
\end{aligned}
$$

and continuity of $L(\cdot)$ follows by

$$
|L(v)| = \int_\Gamma gv\; ds \leq \|g\|_{L_2(\Gamma)} \|v\|_{L_2(\Gamma)} \leq \|g\|_{L_2(\Gamma)} C \|v\|_{H^1(\Omega)},
$$

so that $\kappa_3 = C\|g\|_{L_2(\Gamma)}$.

Problem 3:

(a) Section 21.3

(b) Section 21.3

(c) Section 8.2.1

(d) Section 8.2.2

Problem 4:

Section 16.3

Problem 5:

(a) Section 18.2.3

(b) Section 18.3

(c) Section 18.3

(d) See lecture notes from lectures 6, slides avaliable at:
`http://www.nada.kth.se/kurser/kth/2D1260/lectures/lecture-6.pdf`

(e) To derive sharp estimates of the error in other norms than the energy norm, for general adaptive algorithms.
((See lecture notes from lectures 7, slides avaliable at:
`http://www.nada.kth.se/kurser/kth/2D1260/lectures/lecture-7.pdf`)