Hints and solutions to problems in course book: Computational Differential Equations

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If the problem has been solved at an excercise session this is indicated, see your notes from that session.

If nothing else is indicated the following conventions are used:

$$\|w\| \equiv \|w\|_{L_2(\Omega)}$$

Chapter 4

4.21: Move absolute values inside integral:

$$|(f,g)| = |\int_a^b f(x)\ g(x)\ dx| \leq \int_a^b |f(x)\ g(x)|\ dx = \int_a^b |f(x)|\ |g(x)|\ dx$$

$$\leq \int_a^b \max_{x\in(a,b)} |f(x)|\ |g(x)|\ dx = \max_{x\in[a,b]} |f(x)| \int_a^b |g(x)|\ dx = \|f\|_{L_\infty(a,b)} \|g\|_{L_1(a,b)}$$

4.22: Prove conditions for a norm at p.63. First condition is given by continuity of the function: the only function f such that $\|f\|_{L_p(a,b)} = 0$ for $p = 1, 2, \infty$ is $f = 0$. We then prove the second and third conditions:

$$\|\alpha f\|_{L_1(a,b)} = \int_a^b |\alpha f(x)|\ dx = \int_a^b |\alpha|\ |f(x)|\ dx = |\alpha| \int_a^b |f(x)|\ dx$$

$$\|\alpha f\|_{L_2(a,b)} = (\int_a^b |\alpha f(x)|^2\ dx)^{1/2} = (|\alpha|^2 \int_a^b |f(x)|\ dx)^{1/2} = |\alpha|(\int_a^b |f(x)|\ dx)^{1/2}$$

$$\|\alpha f\|_{L_\infty(a,b)} = \max_{x\in(a,b)} |\alpha f(x)| = \max_{x\in(a,b)} |\alpha|\ |f(x)| = |\alpha| \max_{x\in(a,b)} |f(x)|$$

To prove the third condition, the triangle inequality, we prove some useful inequalities that are valid for general domains $\Omega \subset \mathbb{R}^d$:

Young's inequality:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b \ge 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$$

Proof: $\log(x)$ is concave: $\log(\lambda x_1 + (1-\lambda)x_2) \ge \lambda \log(x_1) + (1-\lambda)\log(x_2)$

$$\log(ab) = \frac{1}{p}\log a^p + \frac{1}{q}\log b^q \le \log(\frac{a^p}{p} + \frac{b^q}{q})$$

Hölder's inequality:

$$\|f\ g\|_{L_1(\Omega)} \le \|f\|_{L_p(\Omega)}\|g\|_{L_q(\Omega)}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Proof: Using Young's inequality, with the notation: $\|f\|_p \equiv \|f\ g\|_{L_p(\Omega)}$

$$\frac{\|f\ g\|_1}{\|f\|_p\|g\|_q} = \frac{1}{\|f\|_p\|g\|_q}\int_\Omega |f\ g|\ dx = \int_\Omega \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}\ dx$$

$$\le \frac{1}{p}\int_\Omega \left(\frac{|f|}{\|f\|_p}\right)^p\ dx + \frac{1}{q}\int_\Omega \left(\frac{|g|}{\|g\|_q}\right)^q\ dx$$

$$= \frac{1}{p}\frac{1}{\|f\|_p^p}\int_\Omega |f|^p\ dx + \frac{1}{q}\frac{1}{\|g\|_q^q}\int_\Omega |g|^q\ dx = \frac{1}{p} + \frac{1}{q} = 1$$

Cauchy-Schwarz inequality: Follows by Hölder's inequality ($p = 2$):

$$|(f,g)| = |\int_\Omega f\ g\ dx| \le \int_\Omega |f\ g|\ dx = \|f\ g\|_1 \le \|f\|_2\|g\|_2$$

Minkowski's inequality:

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

Proof: With $q = p/p - 1$, so that $(p-1)q = p$:

$$|f+g|^p = |f+g|\ |f+g|^{p-1} \leq (|f|+|g|)|f+g|^{p-1} = |f|\ |f+g|^{p-1}+|g|\ |f+g|^{p-1}$$

$$\int_\Omega |f|\ |f+g|^{p-1}\ dx \leq \left(\int_\Omega |f|^p\ dx\right)^{1/p} \left(\int_\Omega |f+g|^{(p-1)q}\ dx\right)^{1/q}$$

$$\int_\Omega |g|\ |f+g|^{p-1}\ dx \leq \left(\int_\Omega |g|^p\ dx\right)^{1/p} \left(\int_\Omega |f+g|^{(p-1)q}\ dx\right)^{1/q}$$

$$\Rightarrow \left(\int_\Omega |f+g|^p\right)^{1/p} = \left(\int_\Omega |f+g|^p\right)^{1-1/q} = \frac{\int_\Omega |f+g|^p\ dx}{\left(\int_\Omega |f+g|^p\right)^{1/q}} \leq \|f\|_p + \|g\|_p$$

Minkowski's inequality gives the triangle inequality for the norms $\|\cdot\|_p$ with $p = 1, 2$, and for $\|\cdot\|_\infty$ we get the triangle inequality by the triangle inequality of the absolute value.

4.24:

$$\|\sin \pi x\|_{L_\infty(0,1)} = \sin\frac{\pi}{2} = 1$$

$$\|\sin \pi x\|_{L_1(0,1)} = \int_0^1 |\sin \pi x|\ dx = \int_0^1 \sin \pi x\ dx = [\frac{-\cos \pi x}{\pi}] = \frac{2}{\pi}$$

$$\|\sin \pi x\|^2_{L_2(0,1)} = ... = \frac{1}{2} - \frac{\sin 2}{4} \ \Rightarrow\ \|\sin \pi x\|_{L_2(0,1)} = ... = \sqrt{\frac{1}{2} - \frac{\sin 2}{4}}$$

$$\|x^2\|_{L_\infty(0,1)} = ... = 1$$

$$\|x^2\|_{L_1(0,1)} = \int_0^1 |x^2|\ dx = \int_0^1 x^2\ dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\|x^2\|_{L_2(0,1)} = \left(\int_0^1 |x^2|^2\ dx\right)^{1/2} = ... = \frac{1}{\sqrt{5}}$$

$$\|e^x\|_{L_\infty(0,1)} = e^1 = e$$

$$\|e^x\|_{L_1(0,1)} = \int_0^1 |e^x|\ dx = \int_0^1 e^x\ dx = ... = e - 1$$

$$\|e^x\|_{L_2(0,1)} = \left(\int_0^1 |e^x|^2 \, dx\right)^{1/2} = ... = \frac{e^2-1}{2}$$

4.25: The function $f(x) = x^{-s}$ is in the space $L_p(0,1)$ if $\|f\|_{L_p(\Omega)} < \infty$.

$$\int_\epsilon^1 x^{-s} \, dx = \left[\frac{x^{1-s}}{1-s}\right]_\epsilon^1 = \frac{1-\epsilon^{1-s}}{1-s}$$

$\epsilon \to 0 \; \Rightarrow \frac{1-\epsilon^{1-s}}{1-s} \to \infty$ if $s \geq 1$. That is, $f(x) = x^{-s} \in \|f\|_{L_1(0,1)}$ if $s < r = 1$. Similarly, $f(x) = x^{-s} \in \|f\|_{L_2(0,1)}$ if $s < r = 0.5$.

Chapter 5

5.8: Use Lagrange's formula (5.1):

$$\pi_3 f(x) = \sin(0)\lambda_0(x) + \sin(\pi/6)\lambda_1(x) + \sin(\pi/4)\lambda_2(x) + \sin(\pi/3)\lambda_3(x)$$

$$= \frac{1}{2}\lambda_1(x) + \frac{1}{\sqrt{2}}\lambda_2(x) + \frac{\sqrt{3}}{2}\lambda_3(x)$$

$$\lambda_1(x)\frac{(x-0)(x-\pi/4)(x-\pi/3)}{(\pi/6-0)(\pi/6-\pi/4)(\pi/6-\pi/3)}$$

$$\lambda_2(x)\frac{(x-0)(x-\pi/6)(x-\pi/3)}{(\pi/4-0)(\pi/4-\pi/6)(\pi/4-\pi/3)}$$

$$\lambda_3(x)\frac{(x-0)(x-\pi/6)(x-\pi/4)}{(\pi/3-0)(\pi/3-\pi/6)(\pi/3-\pi/4)}$$

5.9: Excercise session 3. Use definition of λ_i at p.76:

$$\lambda_0 + \lambda_1 = \frac{x-b}{a-b} + \frac{x-a}{b-a} = \frac{a-b}{a-b} = 1$$

$$(\xi_0 - x)\lambda_0(x) + (\xi_1 - x)\lambda_1(x) = (\xi_0 - x)\frac{x-\xi_1}{\xi_0-\xi_1} + (\xi_1 - x)\frac{x-\xi_0}{\xi_1-\xi_0} = 0$$

Use (5.4),(5.5), and the above equalities, to prove the error representation formula.

5.13:

$$\lambda_0' + \lambda_1' = \frac{1}{a-b} + \frac{1}{b-a} = 0$$

$$(\xi_0 - x)\lambda_0'(x) + (\xi_1 - x)\lambda_1'(x) = \frac{x_0 - x}{\xi_0 - \xi_1} + \frac{\xi_1 - x}{\xi_1 - \xi_0} = 1$$

5.14: See lecture notes from Lecture 4.

5.17: Excercise session 3. Use the coordinate transformation $x = \xi b + (1-\xi)a$, with $\xi \in (0,1)$ and $x \in (a,b)$. This gives that $dx = (b-a)d\xi$, and $\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} = (b-a)\frac{\partial f}{\partial x}$.

$$\|f\|_{L_p(0,1)} = \left(\int_0^1 |f(\xi)|^p \; d\xi\right)^{1/p} = \left(\int_a^b |f(x)|^p \; (b-a)dx\right)^{1/p} = (b–a)^{1/p}\|f\|_{L_p(a,b)}$$

$$\|f''\|_{L_p(0,1)} = \left(\int_0^1 |f''(\xi)|^p \; d\xi\right)^{1/p} = \left(\int_a^b |(b-a)^2 f''(x)|^p \; (b-a)dx\right)^{1/p}$$
$$= (b-a)^{(2p+1)/p}\|f''\|_{L_p(a,b)}$$

Thus,

$$\|f - \pi_i f\|_{L_p(0,1)} \le \|f''\|_{L_p(0,1)}$$
$$\Rightarrow \; (b-a)^{1/p}\|f - \pi_i f\|_{L_p(a,b)} \le (b-a)^{2+1/p}\|f''\|_{L_p(a,b)}$$
$$\Rightarrow \; \|f - \pi_i f\|_{L_p(a,b)} \le (b-a)^2\|f''\|_{L_p(a,b)}$$

5.23: For sub-interval $I_i = (x_{i-1}, x_i)$, choose for example the midpoint $\bar{x}_i = (x_{i-1} + x_i)/2$, and then use the second degree Lagrange polynomial defined on page p.76.

5.24:

$$\varphi_0(x) = \frac{x_1 - x}{x_1 - x_0}, \qquad \varphi_{m+1}(x) = \frac{x - x_m}{x_{m+1} - x_m}$$

Chapter 6

6.2: Use (6.4). You do not have to solve the resulting systems of equations. The case $q = 2$:

$$a_{11} = \frac{1}{1+1} - \frac{1}{1+1+1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$a_{12} = \frac{2}{2+1} - \frac{1}{2+1+1} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$a_{21} = \frac{1}{1+2} - \frac{1}{1+2+1} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$a_{22} = \frac{2}{2+2} - \frac{1}{2+2+1} = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

$$b_1 = \frac{1}{1+1} = \frac{1}{2}$$

$$b_2 = \frac{1}{2+1} = \frac{1}{3}$$

$$\begin{bmatrix} \frac{1}{6} & \frac{3}{4} \\ \frac{1}{12} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

6.8:

$$a_{i-1,i} = \int_{x_{i-1}}^{x_i} \frac{-1}{h_i} \frac{1}{h_i} \, dx = \frac{-1}{h_i^2} \int_{x_{i-1}}^{x_i} dx = \frac{-1}{h_i}$$

6.9: $a_{ii} = 2/h$, $a_{i-1,i} = a_{i,i+1} = -1/h$, and all other entries zero.

6.10: Use the definition of the basis function $\varphi_i(x)$.

6.11: Use Problems 6.9 and 6.10.

6.12: All function in V_h can be written as a linear combination of the basisfunctions: $v \in V_h \;\Rightarrow\; v(x) = \sum_{i=1}^m \eta_i \varphi_i(x), \quad v'(x) = \sum_{i=1}^m \eta_i \varphi_i'(x)$

$$0 \geq (v'(x))^2 = \sum_{i=1}^m \eta_i \varphi_i'(x) \sum_{j=1}^m \eta_j \varphi_j'(x) = \sum_{i,j=1}^m \eta_i \eta_j \varphi_i'(x) \varphi_j'(x) = \sum_{i,j=1}^m \eta_i a_{ij} \eta_j = \eta^T A \eta$$

If not $\eta_i = 0$ for all $i = 1, 2, ..., m$.

6.14: Excercise session 1. Use partial integration in equation (6.19).

Chapter 8

8.6 Similarly for a_{ii-1} and b_i.

$$a_{ii} = \int_0^1 a\varphi_i'\varphi_i' \, dx = \int_{x_{i-1}}^{x_i} (1+x)\frac{1}{h}\frac{1}{h} \, dx + \int_{x_i}^{x_{i+1}} (1+x)\frac{-1}{h}\frac{-1}{h} \, dx$$

$$= \frac{1}{h^2}\left[x + \frac{x^2}{2}\right]_{x_{i-1}}^{x_i} + \frac{1}{h^2}\left[x + \frac{x^2}{2}\right]_{x_i}^{x_{i+1}}$$

$$= \frac{1}{h^2}(h + h\frac{x_{i-1}+x_i}{2}) + \frac{1}{h^2}(h + h\frac{x_i + x_{i+1}}{2})$$

$$= \frac{1}{h}(2 + \frac{x_{i-1}+x_i}{2} + \frac{x_i+x_{i+1}}{2})$$

$$a_{i-1i} = \int_0^1 a\varphi_i'\varphi_{i-1}' \, dx = \int_{x_{i-1}}^{x_i} (1+x)\frac{1}{h}\frac{-1}{h} \, dx = \frac{-1}{h^2}\left[x + \frac{x^2}{2}\right]_{x_{i-1}}^{x_i}$$

$$= \frac{-1}{h^2}(h + h\frac{x_{i-1}+x_i}{2}) = \frac{-1}{h}(1 + \frac{x_{i-1}+x_i}{2})$$

8.7: Yes, the stiffness matrix is still symmetric, positive-definit and tridiagonal.

8.9:

$$g_z'(x) = \begin{cases} 1-z & 0 \le x \le z \\ -z & z \le x \le 1 \end{cases}$$

$$\int_0^1 g_z'(x)e'(x) \, dx = \int_0^z (1-z)e'(x) \, dx + \int_z^1 (-z)e'(x) \, dx$$

$$= (1-z)\int_0^z e'(x) \, dx - z\int_z^1 e'(x) \, dx$$

$$= (1-z)(e(z) - e(0)) - z(e(1) - e(z)) = e(z)$$

$$e(z) = \int_0^1 g_z'(x)e'(x) \, dx = -\int_0^1 g_z(x)u''(x) \, dx + [g_z(x)u'(x)]_0^1$$

$$= -\int_0^1 g_z(x)(u''(x) - U''(x))\ dx = \int_0^1 g_z(x)(f(x) + U''(x))\ dx$$

$= \int_0^1 g_z(x)R(x)\ dx = 0$ if $g_z \in V_h$,
and $g_z \in V_h$ if and only if $z = x_j$ (with x_j a node in the mesh of V_h)

8.10:

$$-\int_0^1 g_z''(x)v(x)\ dx = \int_0^1 g_z'(x)v'(x)\ dx - [g_z'(x)v(x)]_0^1$$

$$= (1-z)\int_0^z v'(x)\ dx - z\int_z^1 v'(x)\ dx - (-zv(1) - (1-z)v(0))$$

$$= (1-z)(v(z)-z(0)) - z(v(1)-v(z)) + (zv(1)+(1-z)v(0)) = v(z) \Rightarrow -g_z''(x) = \delta_z(x)$$

8.11: Excercise session 3.

8.12: Excercise session 3. Partial integration of (8.13), then use the Robin condition (8.12).

8.13: At the left end-node $x_0 = 0$ there is a trial basis function but no test basis function. Integrating over the element $I_i = [x_0, x_1]$, we get the equation

$$\xi_0 - \xi_1 = 0$$

and for the rest of the unknowns ξ_i we get equations:

$$-\xi_{i-1} + 2\xi_i - \xi_{i+1}$$

With M interior nodes we get a $M+1 \times M+1$ system with $i = 0, 1, 2, ..., M$.

8.16:

8.17: Excercise session 5.

$$|\int_0^1 av'w'\ dx| = |\int_0^1 (a^{1/2}v')(a^{1/2}w')\ dx| \leq \|v'\|_a\|w'\|_a$$

$$|\int_0^1 v'w'\ dx| = |\int_0^1 (a^{1/2}v')(a^{-1/2}w')\ dx| \leq \|v'\|_a\|w'\|_{-a}$$

8.18:
$v(x)$ for $x \in (0,1)$ and with $v(0) = 0$:

$$v(y) = \int_0^y v'(x)\ dx = \int_0^y a^{-1/2}(a^{1/2}v'(x))\ dx$$

$$\leq (\int_0^y a^{-1}\ dx)^{1/2}(\int_0^y a(v')^2(x)\ dx)^{1/2} \quad 0 < y < 1$$

If $\int 0^1 a^{-1}\ dx < \infty \Rightarrow$

$$|v(y)| \leq (\int_0^y a^{-1}\ dx)^{1/2}(\int_0^y a(v')^2(x)\, dx)^{1/2} \leq \|1\|_{a^{-1}}\|v\|_E \Rightarrow \|v\|_\infty \leq \|1\|_{a^{-1}}\|v\|_E$$

8.20:

$$\|hR(U)\|_{a^{-1}} = \|ha^{-1/2}(f + (aU')'\| = \|ha^{-1/2}(-(au')' + (aU')')\|$$

$$= \|ha^{-1/2}(-a'u' - au'' + a'U')\| = \|ha^{1/2}(\frac{a'}{a}(U' - u') + (-u'')\|$$

$$\leq C\|u' - U'\|_a + \|hu''\|_a \leq C\|u' - \pi_h u'\|_a + \|hu''\|_a \leq (CC_i + 1)\|hu''\|_a$$

8.21:

8.22:
The a priori error estimate is the same since the same Galerkin orthogonality holds for the weak formulation:

$$\int_0^1 au'v' \; dx = \int_0^1 fv \; dx + g_1 v(1)$$

For the a posteriori error estimate, note that

$$g_1(e(1) - \pi_h e(1)) = 0$$

8.23:

8.37:

8.40:

8.41:

Chapter 9

9.4: Excercise session 4.

9.5: Excercise session 4.

9.10: Excercise session 5.

Chapter 10

10.18: Excercise session 4.

10.21: Excercise session 4.

Chapter 13

13.30 Excercise session 1. Use (13.12) with $w = \dfrac{\partial w}{\partial x_i}$

Chapter 14

14.9 $(\phi_i, \phi_j) = \int_\Omega \phi_i(x)\phi_j(x) \; dx$; is zero for all i, j so that that the support of ϕ_i and ϕ_j do not overlap.

Chapter 15

15.14: Excercise session 1.

15.16: Excercise session 1.

15.19: Excercise session 6.

15.20: Excercise session 6.

15.21: Excercise session 6.

15.44:

15.45: Excercise session 1.

15.48:

15.49: Excercise session 5.

Chapter 16

16.14 Excercise session 4.

Chapter 21

21.1 Excercise session 2. Bilinear and linear forms for equations (8.1), (8.2), and (15.18):

$$(8.1): \quad a(u,v) = \int_0^1 u'v' + cuv \; dx \quad L(v) = \int_0^1 fv \; dx$$

$$(8.2): \quad a(u,v) = \int_0^1 au'v' \; dx \quad L(v) = \int_0^1 fv \; dx$$

$$(15.18): \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \; dx \quad L(v) = \int_\Omega fv \; dx$$

21.2 For symmetric bilinear forms we define the energy norm as

$$\|w\|_a = a(w,w)^{1/2}$$

for which we have the Schwarz inequality:

$$|a(v,w)| \leq \|v\|_a \|w\|_a$$

Proof of equation (21.22):

$$\|u-U\|_a^2 = a(u-U,u-U) = a(u-U,u-U) + a(u-U,U-v)$$
$$= a(u-U,u-v) \le \|u-U\|_a \|u-v\|_a$$

Proof of analog of (21.5): bilinear form $a(\cdot,\cdot)$ is elliptic, and linear form $L(\cdot)$ is continuous, which gives:

$$\kappa_1 \|U\|_V \le a(U,U) = L(U) \le \kappa_3 \|U\|_V$$

21.8(a) Excercise session 2. Let $V = \{v \in H^1(0,1) : \ v(0) = 0\}$,

$$\|w\|_V^2 = \|w\|_{H^1(0,1)}^2 = \int_0^1 |w'|^2 + w^2 \ dx = \|w'\|^2 + \|w\|^2$$

Multiply by $v \in V$, partial integration gives variational form:

$$\int_0^1 u'(x)v'(x) \ dx - [u'(x)v(x)]_0^1 + \alpha \int_0^1 u(x)v(x) \ dx = \int_0^1 f(x)v(x) \ dx$$

Boundary conditions gives:

$$\int_0^1 u'(x)v'(x) \ dx + \alpha \int_0^1 u(x)v(x) \ dx = \int_0^1 f(x)v(x) \ dx$$

Let $a(u,v) = \int_0^1 u'(x)v'(x) \ dx + \int_0^1 \alpha u(x)v(x) \ dx$ and $L(v) = \int_0^1 f(x)v(x) \ dx$.
For $\alpha = 1$ we have $\kappa_1 = 1$:

$$\|w\|_V^2 = \|w'\|^2 + \|w\|^2 = a(w,w)$$

For $\alpha = 0$ we have $\kappa_1 = \sqrt{3}$, using Poincare-Friedrichs inequality (21.17):

$$\|w\|_V^2 = \|w'\|^2 + \|w\|^2 \le \|w'\|^2 + 2\|w'\|^2 = 3\|w'\|^2 = 3a(w,w)$$

That is, $a(\cdot,\cdot)$ is elliptic for $\alpha = 0, 1$. Continuity of $a(\cdot,\cdot)$ with $\kappa_2 = 1$ follows by Cauchy-Schwarz inequality, and since for $v_i, w_i > 0$:

$$v_1 w_1 + v_2 w_2 = (v_1, v_2) \cdot (w_1, w_2) \le (v_1^2 + v_2^2)^{1/2} (w_1^2 + w_2^2)^{1/2}$$

For $\alpha = 1$:

$$a(v,w) = \int_0^1 v'(x)w'(x)\ dx + \int_0^1 v(x)w(x)\ dx \le \|v'\|\ \|w'\| + \|v\|\ \|w\|$$

$$\le (\|v'\|^2 + \|v\|^2)^{1/2}(\|w'\|^2 + \|w\|^2)^{1/2} = \|v\|_V\|w\|_V$$

and for $\alpha = 0$:

$$a(v,w) = \int_0^1 v'(x)w'(x)\ dx\ dx \le \|v'\|\ \|w'\| \le \|v'\|\ \|w'\| + \|v\|\ \|w\|$$

$$\le (\|v'\|^2 + \|v\|^2)^{1/2}(\|w'\|^2 + \|w\|^2)^{1/2} = \|v\|_V\|w\|_V$$

Continuity of $L(\cdot)$ with $\kappa_2 = \|f\|$ follows by Cauchy-Schwarz inequality:

$$L(v) = \int_0^1 f(x)v(x)\ dx \le \|f\|\ \|v\| \le \|f\|\ \|v\|_V = \|f\|\ (\|v\|^2)^{1/2} \le \|f\|\ \|v\|_V$$

21.8(b) Choose $V = \{v : \int_0^1 v'(x)^2\ dx + v(0)^2 + v(1)^2 < \infty\}$

$$\|w\|_V^2 = \int_0^1 w'(x)^2\ dx + w(0)^2 + w(1)^2$$

Multiply by $v \in V$, partial integration gives variational form:

$$\int_0^1 u'(x)v'(x)\ dx - [u'(x)v(x)]_0^1 = \int_0^1 f(x)v(x)\ dx$$

Boundary conditions gives:

$$\int_0^1 u'(x)v'(x)\ dx + u(0)v(0) + u(1)v(1) = \int_0^1 f(x)v(x)\ dx$$

Let

$$a(u,v) = \int_0^1 u'(x)v'(x)\ dx + u(0)v(0) + u(1)v(1)$$

and

$$L(v) = \int_0^1 f(x)v(x)\ dx$$

Ellipticity of $a(\cdot,\cdot)$ with $\kappa_1 = 1$ is given by

$$\|w\|_V^2 = \|w'\|^2 + w(0)^2 + w(1)^2 = a(w,w)$$

Continuity of $a(\cdot,\cdot)$ with $\kappa_2 = 1$ is given by

$$a(v,w) = \int_0^1 v'(x)w'(x)\,dx + v(0)w(0) + v(1)w(1) \le \|v'\|\,\|w'\| + v(0)w(0) + v(1)w(1)$$

$$= (\|v'\|, v(0), v(1)) \cdot (\|w'\|, w(0), w(1))$$

$$\le (\|v'\|^2 + v(0)^2 + v(1)^2)^{1/2}(\|w'\|^2 + w(0)^2 + w(1)^2)^{1/2} = \|v\|_V\|w\|_V$$

Continuity of $L(\cdot)$ with $\kappa_3 = 1$:

$$L(v) = \int_0^1 f(x)v(x)\,dx \le \|f\|\,\|v\| \le \|f\|(\|v\|^2 + v(0)^2 + v(1)^2)^{1/2} = \|f\|\,\|v\|_V$$

21.9 Excercise session 5. Multiply by a test function v and use integration by parts:

$$\int_0^1 u''(x)v''(x)\,dx + \left[u'''(x)v(x)\right]_0^1 - [u''(x)v'(x)]_0^1 = \int_0^1 f(x)v(x)\,dx$$

(a) $u(0) = u'(0) = u(1) = u'(1) = 0$

$$V = \{v \in H^2(0,1) : v(0) = v'(0) = v(1) = v'(1) = 0\}$$

$$H^2(0,1) = \{v : \int_0^1 (v(x)'')^2\,dx + \int_0^1 (v(x)')^2\,dx + \int_0^1 (v(x))^2\,dx < \infty\}$$

$$\|w\|_V^2 = \int_0^1 (w''(x))^2 + (w'(x))^2 + (w(x))^2\,dx$$

$$a(u,v) = \int_0^1 u''(x)v''(x)\,dx, \quad L(v) = \int_0^1 f(x)v(x)\,dx$$

Ellipticity of $a(\cdot,\cdot)$ with $\kappa_1 = \sqrt{7}$ follows by Poincare's inequality (21.17) for v and v':

$$\|v\|_V^2 = \int_0^1 (v''(x))^2 + (v'(x))^2 + (v(x))^2 \; dx$$

$$\leq \int_0^1 (v''(x))^2 + 2(v''(x))^2 + 2(v'(x))^2 \; dx$$

$$\leq \int_0^1 (v''(x))^2 + 2(v''(x))^2 + 4(v''(x))^2 \; dx = 7a(v,v)$$

Continuity of $a(\cdot,\cdot)$ with $\kappa_2 = 1$ follows by Cauchy-Schwarz inequality:

$$a(v,w) = \int_0^1 v''(x)w''(x) \; dx \leq \|v''\| \; \|w''\| \leq \|v\|_V \; \|w\|_V$$

Continuity of $L(\cdot)$ with $\kappa_3 = \|f\|$ follows by Cauchy-Schwarz inequality:

$$L(v) = \int_0^1 f(x)v(x) \; dx \leq \|f\| \; \|v\| \leq \|f\| \; \|v\|_V$$

(b) $u(0) = u''(0) = u'(1) = u'''(1) = 0$

$$V = \{v \in H^2(0,1) : v(0) = v''(0) = v'(1) = v'''(1) = 0\}$$

$$\|w\|_V^2 = \int_0^1 (w''8x))^2 + (w'(x))^2 + (w(x))^2 \; dx$$

$$a(u,v) = \int_0^1 u''(x)v''(x) \; dx, \quad L(v) = \int_0^1 f(x)v(x) \; dx$$

Ellipticity and continuity shown similar to (a).

(c) $u(0) = -u''(0) + u'(0) = 0, \quad u(1) = u''(1) + u'(1) = 0$

$$V = \{v : \int_0^1 (v''(x))^2 \; dx + \int_0^1 (v(x))^2 \; dx + (v'(0))^2 + (v'(1))^2 < \infty, \; v(0) = v(1) = 0\}$$

$$\|w\|_V^2 = \int_0^1 (w''(x))^2 + (w'(x))^2 + (w(x))^2 \; dx$$

$$a(u,v) = \int_0^1 u''(x)v''(x) \; dx, \quad L(v) = \int_0^1 f(x)v(x) \; dx$$

Ellipticity and continuity follows by Cauchy-Schwarz inequality and the inequality $v_1w_1 + v_2w_2 + v_3w_3 \leq (v_1 + v_2 + v_3)(w_1 + w_2 + w_3)$ for $v_i, w_i > 0$.

21.11 $f \in H^{-1}(\Omega)$ means that $\|f\|_{H^{-1}(\Omega)} < \infty$, with

$$\|f\|_{H^{-1}(\Omega)} \equiv \sup_{v \in H_0^1(\Omega)} \frac{(f, v)}{\|v\|_V}$$

This gives that

$$\frac{(f, v)}{\|v\|_V} \leq \|f\|_{H^{-1}(\Omega)} \quad \Rightarrow \quad (f, v) \leq \|f\|_{H^{-1}(\Omega)} \|v\|_V$$

with $\|\cdot\|_V = \|\cdot\|_{H^1(\Omega)}$. Thus

$$L(v) = \int_\Omega fv \; dx + \int_\Gamma gv \; ds \leq \|f\|_{H^{-1}(\Omega)}\|v\|_V + \|g\|_{L_2(\Gamma)}\|v\|_{L_2(\Gamma)}$$

$$\leq \|f\|_{H^{-1}(\Omega)}\|v\|_V + \|g\|_{L_2(\Gamma)}C\|v\|_{H^1(\Omega)} \leq (\|f\|_{H^{-1}(\Omega)} + C\|g\|_{L_2(\Gamma)})\|v\|_V$$

by the trace inequality (21.20):

$$\|v\|_{L_2(\Gamma)} \leq C\|v\|_{H^1(\Omega)}$$

21.12 Excercise session 2. Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega fv \; dx, \quad \forall v \in \hat{V}$$

with

$$V = \{v \in H^1(\Omega) : \; v = g \text{ on } \Gamma\}, \quad \hat{V} = H_0^1(\Omega)$$

21.13 Excercise session 2. Similar to 21.8, but use Green's formula instead of integration by parts. For the case of $\alpha = \sigma = 0$ we cannot use Poincare-Friedrich's inequality (21.18), we cannot say anything about $\|u\|_{L_2(\Gamma)}$ since we only have a Neumann boundary condition $\partial_n u = g$ on Γ. Thus the conditions for Lax-Milgram are not satisfied. It is easy to see that indeed we do not have a unique solution to this problem, since for any solution $u(x)$, the solution plus a constant $u(x) + C$ is also a solution.

21.14 We prove ellipticity of $a(\cdot,\cdot)$ using

$$\int_\Omega (\beta \cdot \nabla v \; v + \alpha v^2) \; dx \geq 0$$

and the Poincare-Friedrich inequality (21.18):

$$a(w,w) = \int_\Omega \epsilon \nabla w \cdot \nabla w + \beta \cdot \nabla w \; w + \alpha w \; w \; dx \geq \epsilon \int_\Omega \nabla w \cdot \nabla w \; dx$$

$$\|w\|_V^2 = \|w\|^2 + \|\nabla w\|^2 \leq (C+1)\|\nabla w\|^2 \leq \frac{C+1}{\epsilon} \; a(w,w)$$

This means that the constant $\kappa_1 = \dfrac{\epsilon}{C+1}$ decreases as ϵ decreases.

21.17 Use Poincare-Friedrich inequality for each vector component, and use the norm

$$\|v\|_V = \int_\Omega \sum_{i=1}^{3} |\nabla v_i| \; dx$$