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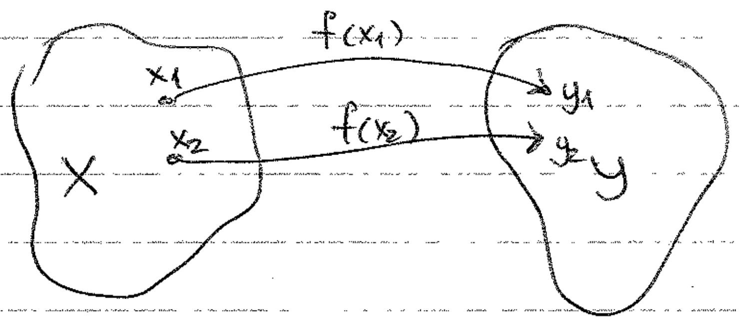
Lecture 1

* Poisson 1D
* boundary conditions
* weak formulations
* Galerkin method
* piecewise polynomials 1D

Some basic consepts:



Def. Mapping from X to Y: each element of X is mapped to Y by function f.
It is usually denoted by $f: X \to Y$

Def. Linear space: the following operations are defined: 1. addition; 2. multiplication to ~~any or~~ a number; 3. addition + multiplication.

Def. Normed linear space. to each element $x \in R$ $\exists$ a number $\|x\| \geq 0$ s.t.

1) $\|x\| = 0$ iff $x = 0$
2) $\|\alpha x\| = |\alpha| \, \|x\|$
3) $\|x+y\| \leq \|x\| + \|y\|$

$L_p(\Omega)$ space — the space with inner product, the following integral exist

$$\left| \int_\Omega |f(x)|^p dx \right| < \infty ;$$

$$(f,g) = \int_\Omega fg\,dx ,$$

$$\|f\|_{L_p} = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}$$

$$L_2(\Omega) \overset{def}{=} \left\{ v : \int_\Omega v^2 d\Omega < \infty \right\}$$ L_2 space

$$\|f\|_{L_2(\Omega)} = \left(\int_\Omega f^2 \right)^{\frac{1}{2}}$$ — norm

$$(f,g) = \int_\Omega fg\,dx$$ — scalar product

$$|(f,g)| \le \|f\|_{L_2(\Omega)} \|g\|_{L_2(\Omega)}$$ — Cauchy–Schwartz

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$$H^1(\Omega) = \left\{ v : \int_\Omega (|\nabla v|^2 + v^2)\, d\Omega < \infty \right\}$$ — Sobolev space

$$(f,g)_{H^1} = \int_\Omega (\nabla u \cdot \nabla v + uv)\, d\Omega ,$$

$$\|f\|_{H^1} = (u,u)_{H^1}^{1/2}$$

Green's formula: $\int_\Omega \nabla u\, v\, dx = -\int_\Omega u \nabla v\, dx + \int_{\partial\Omega} u\, n\, v\, ds$

in 1D: $\int_a^b u' v\, dx = -\int_a^b u v'\, dx + uv|_a^b$

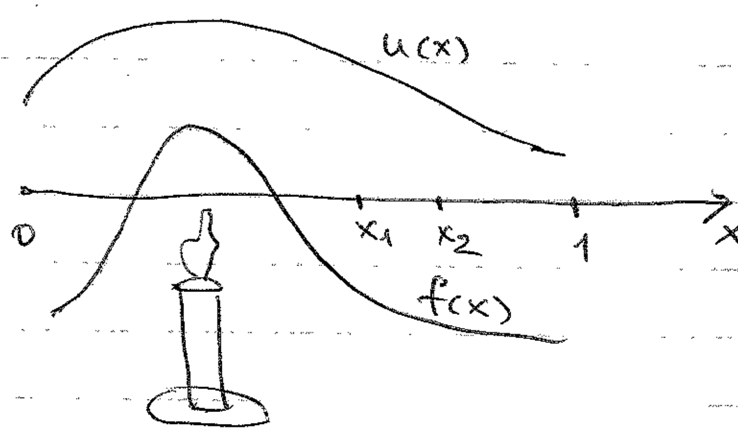
Example. Metric space: $\rho(x,y) = \|x-y\|$

Def. A complete normed space is Banach space

Def. Differential equation: an equation that involves a derivative of a function and the function itself.

Heat conduction in thin wire

Heat conduction in a thin heat-conductivity wire in $[0,1]$, heated by heat source $f(x)$. What is the stationary distribution of the temperature $u(x)$?



$q(x)$ – heat flux in the direction of positive x-axis

Energy conservation: for arbitrary sub-interval $(x_1, x_2) \in (0,1)$ net heat flux through the endpoints = produced heat in (x_1, x_2) per unit time:

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x)\,dx$$

Books: 1. C. Johnson : Numerical solution of PDEs by the FEM

2. Computational differential equations
Eriksson, Estep, Hansbo, Johnson

3. Lecture notes by P. Hansbo:
ex: Beyond the elements of finite elements: General principles for solid and fluid mechanics

By the Fundamental theorem of Calculus

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} q'(x)\,dx$$

from which we conclude that

$$\int_{x_1}^{x_2} q'(x)\,dx = \int_{x_1}^{x_2} f(x)\,dx$$

Since x_1, x_2 are arbitrary + assuming $q'(x), f(x)$ are continuous $\Rightarrow$

$$q'(x) = f(x) \quad \text{for } 0 < x < 1$$

which expresses conservation of energy in differential equation form.

Constitutive relation states: (Fourier's law)

$$q(x) = -a(x)\,u'(x)$$

(Heat flows from warm regions to cold, proportional to a temperature gradient $a(x) > 0$, coeff of heat conductivity)

$$\Rightarrow \quad (-a(x)\,u'(x))' = f(x) \quad \text{for } 0 < x < 1$$

Stationary heat equation.

To define $u(x)$ uniquely we need boundary c

Ex. Homogeneous Dirichlet bc. $u(0)=u(1)=0$
(temp is zero at end points)

$\Rightarrow$ Two-points boundary value problem:

$$\begin{cases} -(au')' = f \\ u(0) = u(1) = 0 \end{cases}$$

Ex. Homogeneous Neumann bc. $q(0) = -a(0)u'(0$
(insulating wire at $x=0$)

Non-homogeneous Dirichlet/Neumann bcs:
$u(0) = u_0$, $q(0) = g_0$ (prescribed temp / flux)

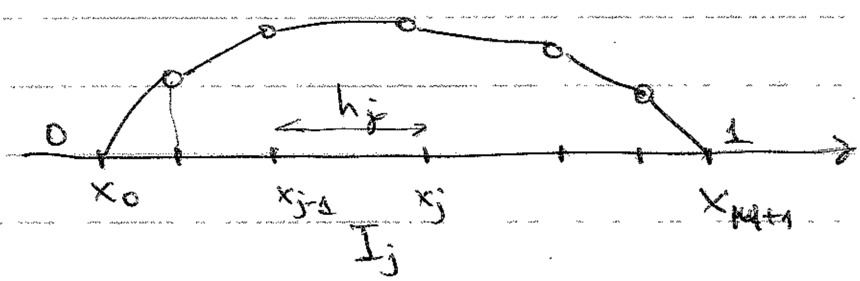
Robin boundary cond. at $x=1$
$(a(1)\, u'(1) + \gamma\,(u(1) - u_1) = g_1$

$\gamma \geq 0$ given boundary / heat conductivity
$\gamma = 0 \Rightarrow$ Neumann $\Rightarrow$ $a(1)u'(1) = g_1$
$\gamma = \infty \Rightarrow$ Dirichlet $u(1) = u_1$
$g_1 = 0 \Rightarrow$ heat flux $-a(1)u'(1)$ is propotional to Temperature difference $u(1) - u_1$
(u_1 - arbitrary temp)

Assume $a \equiv 1$. Find numerical approximation of DE

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Piecewise linear approximation on mesh $\mathcal{T}_h$

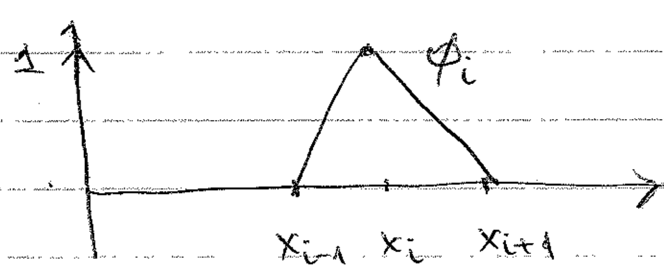


Dewide $[0,1]$ into subintervals $I_j = (x_{j-1}, x_j)$ of length $h_j = x_j - x_{j-1}$; $\mathcal{T}_h$: $0 = x_0 < x_1 < \dots < x_{M+1} = 1$

$V_h = \{$ set of all cont piecewise linear functions on $\mathcal{T}_h$ that are zero at $x=0$, $x=1$ $\}$

V_h - is a finite dimentional vector space of dimention M - degrees of fridom, nodes.

Basic function of V_h: hat functions $\{\varphi_j\}_{j=1}^{M}$



$$\varphi_i(x) = \begin{cases} 0 & x \notin [x_{i-1}, x_{i+1}] \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & x \in [x_i, x_{i+1}] \end{cases}$$

Then, any function $v(x)$ in V_h can be written

$$v(x) = \sum_{j=1}^{M} v(x_j)\,\phi_j(x) = \sum_{j=1}^{M} \xi_j\, \phi_j(x)$$

Construct approximate solution

$$U(x) = \sum_{j=1}^{M} \xi_j\, \phi_j(x) \in V_h$$

Residual: $R(U) = -U'' - f$

if $u(x)$ – is exact sol $\Rightarrow R(u) = 0$.

Galerkine method: Find $U \in V_h$ s.t.

$$\big(R(U), v\big) = 0 \quad \forall v \in V_h$$

(Choose $U \in V_h$ s.t. $R(U)$ orthogonal to all $v \in V_h$)

$(f,g) = \int_0^1 f(x)\,g(x)\,dx$ – L_2 scalar product in $(0,1)$

$$\Rightarrow \big(R(U), v\big) = \int_0^1 R(U)\,v\,dx = \int_0^1 (-U'' - f)\,v\,dx = 0 \Leftrightarrow \int_0^1 -U''v\,dx = \int_0^1 fv\,d$$

Weak form / Variational formulation

$$-\int_0^1 u''v\,dx = \int_0^1 u'v'\,dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 u'v'\,dx$$
$$= \int_0^1 fv\,dx$$

(Since $v \in V_h \Rightarrow v(0) = v(1) = 0$, use part. inegrations)

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du \;;\; \int_a^b fg\,dx = \left[f\int g\,dx\right]_a^b - \int_a^b \left(\int g\,dx\right) df$$

Galerkin FEM: Find $U \in V_h$ such that

(G) $\displaystyle\int_0^1 U'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V_h$

Weak form of DE $(*)$: Find $u \in V$ such-that

(W) $\displaystyle\int_0^1 u'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V$

$V = \{$all functions $v(x)$ s.t. $v(0) = v(1) = 0$ and integrals are well defined$\}$

Galerkin orthogonality property: (W) − (G)

$$\int_0^1 (u' - U')\,v'\,dx = 0 \quad \forall v \in V_h \quad (G.O)$$

(Note : $V_h \in V$)

<u>Norms</u>: measuring the size of a function.

L_2-norm : $\|f\| = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}$

Energy norm: $\|v\|_E = \left(\int_0^1 |v'(x)|^2 dx \right)^{1/2}$

How large is the error $u-U$ in $\|\cdot\|_E$?

$$\|(u-U)'\|^2_{L_2} = \int_0^1 (u-U)'(u-U)'dx = \left[\pm \int_0^1 \overbrace{v'}^{(u-U)'} dx \right]$$

$$= \int_0^1 (u-U)'(u-v)'dx + \int_0^1 (u-U)'\underbrace{(v-U)'}_{\to 0 \ (G.O.)}dx$$

$$= \int_0^1 (u-U)'(u-v)'dx \leq \|(u-U)'\|_{L_2} \|(u-v)'\|_{L_2}$$

[Cauchy-Schwartz ineq $\int_0^1 fg\,dx \leq \|f\|_{L_2[0,1]} \|g\|_{L_2[0,1]}$]

$$\Rightarrow \|(u-U)'\|^2_{L_2} \leq \|(u-U)'\|_{L_2} \|(u-v)'\|_{L_2} \quad \forall v \in V_h$$

$$\|(u-U)'\|_E \leq \|(u-v)'\|_E \quad \forall v \in V_h \Rightarrow$$

Galerkin appr. is optimal in energy norm!

The discrete system of equations

$U(x) = \sum_{j=1}^{M} \xi_j \phi_j(x)$ is determined by $\{\xi_j\}_{j=1}^{M}$

Determine $\{\xi_j\}_{j=1}^{M}$ from (G) !

$$(G) \quad \int_0^1 U'v'\,dx = \int_0^1 f v\,dx \qquad \forall v \in V_h$$

$$\Rightarrow \int_0^1 \Big(\sum_{j=1}^{M} \xi_j \phi_j(x)\Big)' \phi_i'\,dx = \int_0^1 f \phi_i\,dx, \; i=1,\dots,M$$

($\{\varphi_i\}_{i=1}^{M}$ basis of $V_h \Rightarrow$ suff. to check for basis)

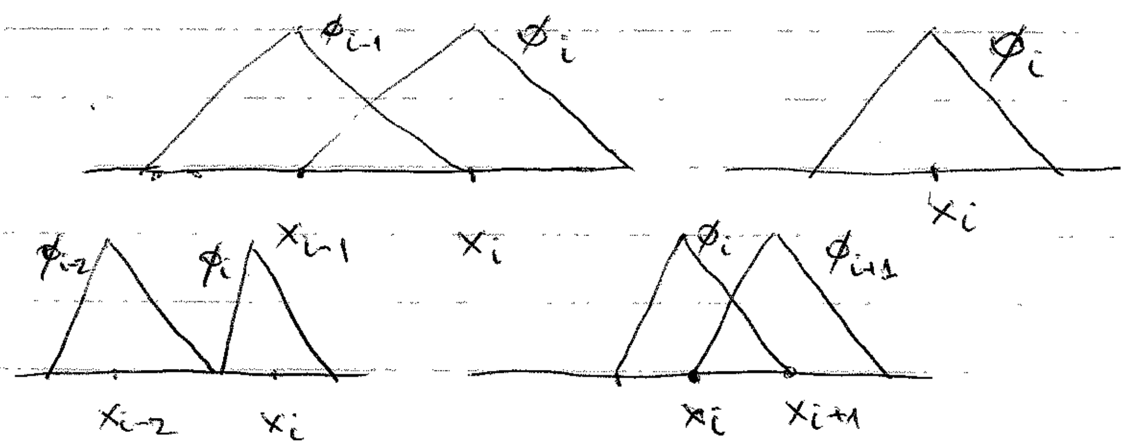
$$\Rightarrow \sum_{j=1}^{M} \xi_j \int_0^1 \phi_j' \phi_i'\,dx = \int_0^1 f \phi_i\,dx, \; i=1,\dots,M$$

It corresponds to linear system of eq's $A\xi = b$ with

$$A = \{a_{ij}\}, \quad b = (b_i), \quad \xi = (\xi_j)$$

$$a_{ij} = \int_0^1 \phi_j' \varphi_i'\,dx, \quad b = \int_0^1 f \phi_i\,dx$$

Matrix A – sparse since all ellements $a_{ij} = 0$ unless $i = j-1$, $i = j$, $i = j+1$ $\Leftrightarrow$ $\varphi_i(x)$ or $\phi_j(x) = 0$ in each subinterval:

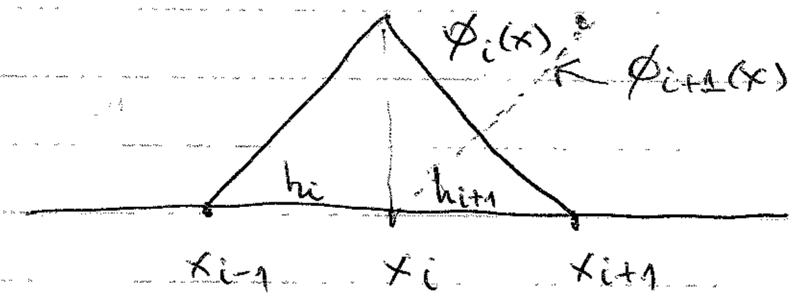


$\phi_{i-2} \cdot \varphi_i = 0$

A is symmetric : $a_{ij} = \int_0^1 \phi_j' \phi_i' \, dx = \int_0^1 \phi_i' \phi_j' \, dx =$

By the definition

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/h_i \, ; & x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/h_{i+1} \, ; & x_i \le x \le x_{i+1} \end{cases}$$



$$\phi_i'(x) = \begin{cases} \frac{1}{h_i} \, ; & x_{i-1} \le x \le x_i \\ -\frac{1}{h_{i+1}} \, ; & x_i \le x \le x_{i+1} \end{cases}$$

$$a_{ii} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_{i+1}}\right)^2 dx = \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

$$a_{i,i+1} = \int_{x_i}^{x_{i+1}} -\frac{1}{h_{i+1}} \frac{1}{h_{i+1}} \, dx = -\frac{1}{h_{i+1}} = a_{i+1,i}$$

$$a_{i-1,i} = \int_{x_{i-1}}^{x_i} \frac{1}{h_i} \cdot \frac{-1}{h_i} \, dx = -\frac{1}{h_i} = a_{i,i-1}$$

$$b_i = \int_{x_{i-1}}^{x_i} f(x) \frac{x - x_{i-1}}{h_i} \, dx + \int_{x_i}^{x_{i+1}} f(x) \frac{x_{i+1} - x_i}{h_{i+1}} \, dx$$

<u>Theorem</u> The stiffness martix A is symetric and positive definite.

Proof: * <u>Symmetric</u>, since

$$A = (a_{ij}) = \overline{\Sigma} \int_0^1 \phi_j' \phi_i' dx = \int_0^1 \phi_i' \phi_j' dx$$

* <u>Positive definite</u>. For arbitrary $\eta_i \neq 0$, $\eta \in \mathbb{R}^M$

$\eta^T A \eta > 0$:

$$= \sum_{i,j=1}^{M} \eta_i a_{ij} \eta_j = \sum_{i,j=1}^{M} \eta_i \int_0^1 \phi_i' \phi_j' dx \; \eta_j =$$

$$= \int_0^1 \left[\sum_{i,j=1}^{M} \eta_i \phi_i' \phi_j' \eta_j \right] dx$$

$$\left[\text{choose } v(x) = \eta_i \phi_i(x) \in V_h \right]$$

$$\Rightarrow \int_0^1 \left[\sum_{i,j=1}^{M} v_i' v_j' \right] dx = \int_0^1 (v')^2 dx > 0$$

unless $v = 0$.

<u>Theorem 2.</u> There exists unique solution $\xi \in \mathbb{R}^M$ of $A\xi = b$

<u>Proof</u>: A is positive definite, nonsingular $\Rightarrow$ existence, uniqueness.

Future reading:

CDE: Ch 1–4 : Background
CDE: Ch 6, 8.1 : Galerkin FEM

Lecture 2

* Poisson equations in 2D
* FEM mesh
* piecewise polynomials 2D
* quadrature ; Affine mapping.

Poisson equations in $\mathbb{R}^2$ with homogeneous Dirichlet bcs:

$$(D) \begin{cases} -\Delta u(x) = f(x) , & x \in \Omega \\ u(x) = 0 , & x \in \Gamma \end{cases}$$

where Ω bounded domain in $\mathbb{R}^2$ with boundary Γ

$$-\Delta u = -\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} , \quad x = (x_1, x_2)$$

"Laplacian of $u(x)$"

Variational formulation

Find $u \in V$ such that

$$(\nabla u, \nabla v) = (f, v) , \quad \forall v \in V$$

where $(w, v) = \int_\Omega w v \, dx$, $(\nabla w, \nabla v) = \int_\Omega \nabla w \cdot \nabla v \, dx$

and $V = \left\{ v : \int_\Omega (|\nabla v|^2 + v^2) \, dx < \infty \text{ and } v = 0 \text{ on } \Gamma \right\}$

(A Hilbert space is a vector space with a scalar product)

A second order PDE

(1) $A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F = 0$

$$Z = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

* (1) is elliptic if $|Z| > 0$

Ex: Poisson: $\nabla^2 u = f$

Laplace $\nabla^2 u = 0$

* (1) is hyperbolic if $|Z| < 0$

Ex: Wave: $\dfrac{\partial^2 \psi}{\partial x^2} = \dfrac{1}{v^2} \dfrac{\partial^2 \psi}{\partial t^2}$; $\dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$

Ex: ~~Hyperbolic~~ conservation laws.

* (1) is parabolic if $|Z| = 0$

Ex: heat eq: $\partial_t u = \kappa \nabla^2 u$ in 2D

$\partial_t u = \kappa \dfrac{\partial^2 u}{\partial x^2}$

L_2 space : $L_2 = \{ v : \int_\Omega |v|^2 dx < \infty \}$

in (W) $f \in L_2(\Omega)$, $\overset{v \in V}{\Rightarrow}$ (f, v) is well defi

if $u, v \in V \Rightarrow (\nabla u, \nabla w)$ is well defined.

Green's formula

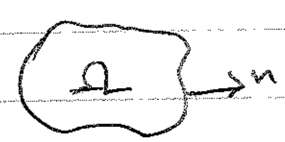
$$\int_\Omega \nabla v \cdot \nabla w \, dx = \int_\Gamma v \, \partial_n w \, ds - \int_\Omega v \Delta w$$

Multiply $-\Delta u = f$ by $v \in V$ and using Green's formula

$$\int_\Omega -\Delta u \, v \, dx = \int_\Omega f v \, dx$$

$$\Rightarrow \int_\Omega \nabla u \nabla v \, dx - \underbrace{\cancel{\int_\Gamma v \, \partial_n u \, ds}}_{\to 0 \ (v \in V \Rightarrow v = 0 \text{ on } \Gamma)} = \int_\Omega f v \, dx$$

$\partial_n u = \nabla u \cdot n = \frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2$ — normal d

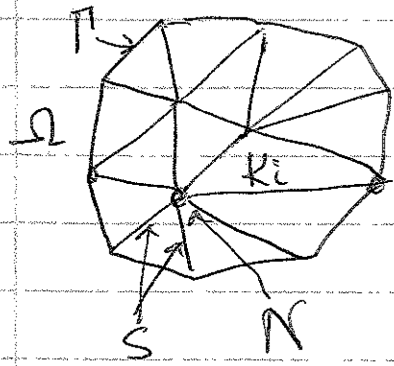


, $n = (n_1, n_2)$

(D) and (W) have the same solution if f continuous, see Ch 21 CDE.

* A triangular mesh of Ω with a polygonal Γ

$\mathcal{T}_h = \{K\}$ - is a subdivision of Ω into nonoverlapping triangles /elements/cells K_i sush that there is no "hanging nodes".



$\mathcal{N}_h = \{N\}$ nodes/vertices of $\mathcal{T}_h$

$S_h = \{S\}$ edges of $\mathcal{T}_h$

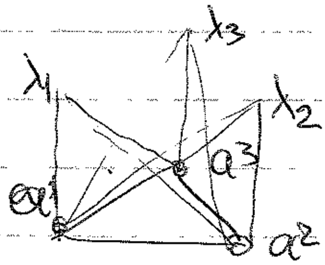
h_K - is a diameter of K (longest side of K)

$h(x)$ - mesh function - p.w. constant :
$h(x) = h_K$ for $x \in K$, $K \in \mathcal{T}_h$

$V_h = \{ v : v$ is contin on Ω, $v|_K \in \mathcal{P}^1(K), K \in \mathcal{T}_h \}$

$\mathcal{P}^1(K)$ - linear functions on K : $v = c_0 + c_1 x_1 + c_2 x_2$
c_i - const

$\lambda_i \in \mathcal{P}^1(K)$, $i = 1,2,3$ - element basic func



$$\lambda_i(a^j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$\{\phi_j(N_i)\}_{j=1}^{M}$ – global basis func's (tent func)

for $N_1, N_2, \dots, N_M$ – nodes $\in \mathcal{N}_h$

$$\varphi_j \in V_h \ ; \quad \phi_j(N_i) = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}, \quad i,j=\overline{1,M}$$

$$\Rightarrow v \in V_h : \quad v(x) = \sum_{i=1}^{M} v(N_i)\,\phi_i(x)$$

Discrete system

find $u \in V$ s.t.

$$(V) \quad (\nabla u, \nabla v) = (f, v), \quad \forall v \in V$$

$$v \in H_0^1 = \left\{ v : \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty, \ v=0 \text{ on } \Gamma \right\}$$

$$V_h = \{ v \in V : v \text{ p.w. linear on } \mathcal{T}_h \}$$

Let $\{N_1, N_2, \dots, N_M\}$ – nodes

$\{\phi_1, \phi_2, \dots, \phi_M\}$ – nodal basis for V_h

Galerkin FEM: Find $U \in V_h$ s.t.

$$(G) \quad (\nabla U, \nabla v) = (f, v) \quad \forall v \in V_h$$

$V_h \in V \Rightarrow$ Galerkin orthogonality $(V)-(G)$

$$(\nabla u - \nabla U, \nabla v) = 0, \quad \forall v \in V_h$$

or $(R(U), \nabla v) = 0$

Discrete system of equations

$U \in V_h \Rightarrow U = \sum_{j=1}^{M} \xi_j \phi_j \quad , \xi_j = U(N_j)$

(G) and choosing $v = \phi_i \Rightarrow$

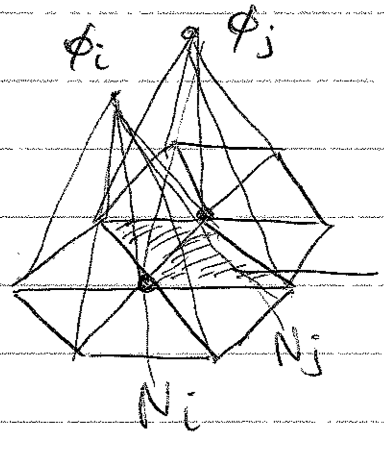
$$\sum_{j=1}^{M} (\nabla\phi_j, \nabla\phi_i)\xi_j = (f, \phi_i) \quad , i = 1, \dots, M$$

$$\Rightarrow \quad A\xi = b$$

$A = (a_{ij}) \quad , b = (b_i) \quad , \xi = (\xi_i)$

$a_{ij} = (\nabla\phi_j, \nabla\phi_i) \quad , b_i = (f, \phi_i)$

A - is sparse



a_{ij} - is nonzero if ϕ_i and ϕ_j have common support $\Rightarrow$

* $i = j$
* N_i and N_j neigh

Assembly algorithm: to compute global matrix A

For all $K \in \mathcal{T}_h$

Compute element stiffness matrix A^K_{ij}

for all $i, j = 1, 2, 3$

Add contribution to global matrix

$a_{ij} \mathrel{+}= A^K_{ij}$

Solving a linear system $A\xi = b$

(1) Direct methods, (LU, ~~Gaussian elim~~)
(2) Iterative solvers (CG, GMRES, MINRES, ...)

A – large & sparse $\Rightarrow$ iterative solvers

Computation of element stiffess

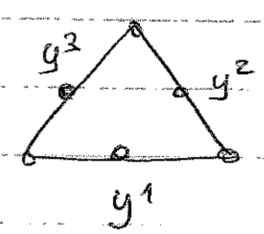
$$A^K_{ij} = \int_K \nabla \phi_j \cdot \nabla \phi_i \, dx \quad , \quad i,j = 1,2,3$$

* Quadrature: If exact calculation of integrals is inefficient or impossible

$$\int_K g(x)\,dx \simeq \sum_{i=1}^{q} g(y^i)\, w_i$$

q – # quadrature points
y_i – quadrature points (nodes)
w_i – quadrature weights

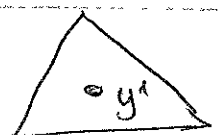
Ex. Midpoint quadrature



$q = 3$, y^i – edge midpoints

$w_i = \frac{|K|}{3}$, $|K|$ – area of K

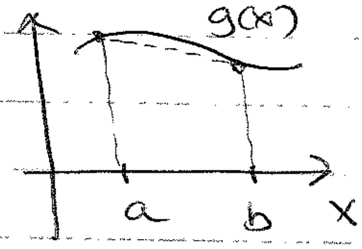
Ex.



$q = 1$, $w_1 = |K|$

Ex. Trapezoidal rule in 1D.

$$\int_a^b g(x)\,dx \simeq (b-a)\,\frac{g(a)+g(b)}{2}$$

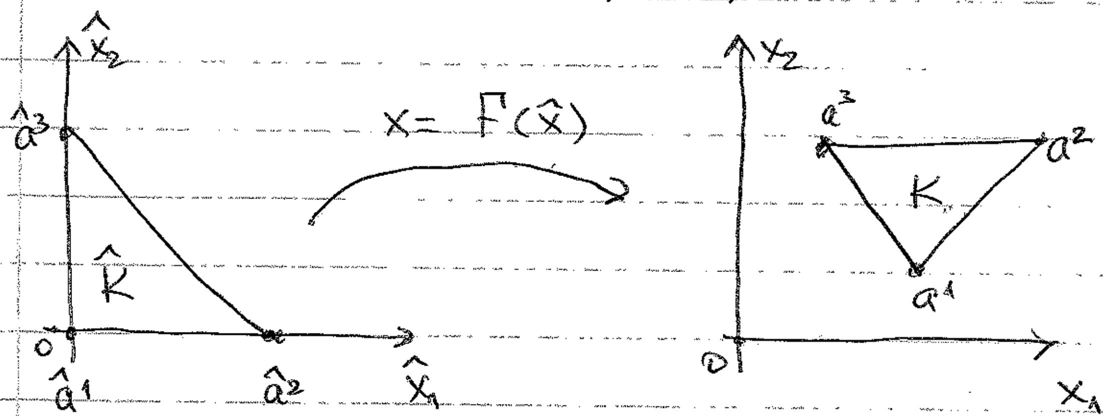


accuracy

exact for $\mathcal{P}^k$

* Affine mapping

It is a common practise to map the triangle K to a reference triangle $\hat{K}$ and do quadrature rule in terms of reference coordinate



$$F(\hat{x}) \overset{def}{=} a^1 \hat{\phi}_1(\hat{x}) + a^2 \hat{\phi}_2(\hat{x}) + a^3 \hat{\phi}_3(\hat{x})$$

where $\hat{\phi}_i(\hat{x})$ – is a nodal basis function of the reference triangle:

$$\hat{\phi}_1(\hat{x}) = 1 - \hat{x}_1 - \hat{x}_2 \ , \quad \hat{\phi}_2(\hat{x}) = \hat{x}_1 \ , \quad \hat{\phi}_3(\hat{x}) = \hat{x}_2$$

* Computation of the element stiffness martix

The local basis on K are given by

$$\phi_j(x) = \hat{\phi}_j(F^{-1}(x)) \ , \quad j = 1,2,3$$

Then

$$(I) \quad a_{ij}^K = \int_K \nabla \phi_i \, \nabla \phi_j \, dx \ , \quad i,j = 1,2,3$$

by the chain rule

$$\frac{\partial \phi_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\hat{\phi}_i(F^{-1}(x))\right) = \frac{\partial \hat{\phi}_i}{\partial \hat{x}_1}\frac{\partial \hat{x}_1}{\partial x_j} + \frac{\partial \hat{\phi}_i}{\partial \hat{x}_2}\frac{\partial \hat{x}_2}{\partial x_j}$$

So that $\nabla \phi_i = J^{-1} \nabla \hat{\phi}_i$, where

$$J^{-1} = \begin{bmatrix} \frac{\partial \hat{x}_1}{\partial x_1} & \frac{\partial \hat{x}_2}{\partial x_1} \\ \frac{\partial \hat{x}_1}{\partial x_2} & \frac{\partial \hat{x}_2}{\partial x_2} \end{bmatrix}$$ Jacobian of F^{-1}.

$$(I) \Rightarrow a_{ij}^{k} = \int_{\hat{K}} (J^{-1}\nabla\hat{\phi}_i)\cdot(J^{-1}\nabla\hat{\phi}_j)\,|\det J|\,d\hat{x}$$

since $dx = |\det J|\, d\hat{x}$

Further reading:

CDE 15.1 : 2D Poisson eqn
CDE 13 : Background
CDE 14.1 - 14.2 : 2D Mesh & p.w. polyn.
CDE 5.5, 14.4 : Quadrature

Lecture 3

* Boundary conditions
* Adaptivity, residual, mesh refinement

Poisson equation in ~~Ω~~ $\mathbb{R}^2$ with non-homogenuous ~~boundary~~ Dirichlet boundary conditions

$$(D) \begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = g & \text{on } \Gamma \end{cases}$$

g - is a given boundary data

<u>Variational form</u>. Find $u \in V_g$ s.t

$$(V) \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V_0$$

where $V_g = \left\{ v : v = g \text{ on } \Gamma \ \& \ \int_\Omega (|\nabla v|^2 + v^2) dx < \infty \right\}$

$V_0 = \left\{ v : v = 0 \text{ on } \Gamma \ \& \ \int_\Omega (|\nabla v|^2 + v^2) dx < \infty \right\}$

V_g - trial test, V_0 - test space

The test space is choosen such that the boundary integral dissappears in (V)

$$(\nabla u, \nabla v) - \cancel{\int_\Gamma (\nabla u \cdot n) \overset{0}{v} ds} = (f, v)$$

Ex. 1D Poisson with ~~non~~-non-homogeneous Dirichlet bc

$$\begin{cases} -u'' = f & \text{on } (0,1) \\ u(0) = g, & u(1) = 0 \end{cases}$$

Variational formulation:

$$\int_0^1 -u''v\,dx = \int_0^1 fv\,dx$$

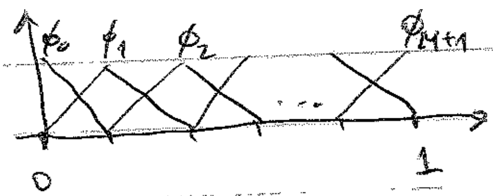
$$\Rightarrow \int_0^1 u'v'dx - [u'v]_0^1 = \int_0^1 fv\,dx, \quad \forall v \in V_0$$

$$V_0 = \{v : \|v\|^2 + \|v'\|^2 < \infty,\ v(0) = v(1) = 0\}$$

$$u \in V_g = \{v : \|v\|^2 + \|v'\|^2 < \infty,\ v(0) = g,\ v(1) = 0\}$$

$$\Rightarrow \text{Find } u \in V_g \text{ s.t. } \int_0^1 u'v'dx = \int_0^1 fv\,dx, \ \forall v \in V_0$$

FEM: find $U \in V_h^g \subset V_g$ s.t. $\int_0^1 U'v'dx = \int_0^1 fv\,dx, \ \forall v \in V_h^0$



$$U = \sum_{j=0}^{M+1} \xi_j \phi_j(x)$$

From the bc: $U(0) = \xi_0 = g$, $U(1) = \xi_{M+1} = 0$

$$\Rightarrow U = \sum_{j=1}^{M} \xi_j \phi_j(x) + g\phi_0(x) + 0 \cdot \phi_{M+1}$$

$$\Rightarrow \underbrace{\sum_{j=1}^{M} \xi_j \int_0^1 \phi_j \phi_i\,dx}_{A\xi} = \underbrace{\int_0^1 f\phi_i\,dx}_{b} - g \int_0^1 \phi_0' \phi_i\,dx, \quad i=1,\dots$$

$$\begin{pmatrix} \int_0^1 \phi_1\phi_1 dx & \cdots \\ & \\ & \\ & \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_M \end{pmatrix} = \begin{pmatrix} \int_0^1 f\phi_1 dx \\ \int_0^1 f\phi_2 dx \\ \vdots \\ \int_0^1 f\phi_M dx \end{pmatrix} - \begin{pmatrix} g\int_0^1 (-\frac{1}{h})\phi_1 dx \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

When computing FEM approximation U, set nodes on the boundary $\mathcal{N}_b$ and interior nodes to $\mathcal{N}_h$:

$$U = \sum_{N_j \in \mathcal{N}_b} \xi_j \phi_j + \sum_{N_j \in \mathcal{N}_h} \xi_j \phi_j$$

where $\xi_j = g(N_j)$ for $N_j \in \mathcal{N}_b$

$\Rightarrow$ The resulting discrete system is

$$\sum_{N_j \in \mathcal{N}_h} \xi_j (\nabla \phi_j, \nabla \phi_i) = (f, \phi_i) \boxed{- \sum_{N_j \in \mathcal{N}_b} g(N_j)(\nabla \phi_j, \nabla \phi_i)}$$

$\forall \; N_i \in \mathcal{N}_h$

Robin and Neumann bc's

$$(D) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \partial_n u + \gamma u = g & \text{on } \Gamma_2 \end{cases} \qquad \Gamma = \Gamma_1 \cup \Gamma_2, \quad \gamma \geq 0$$

Multiply (D) ~~by~~ ~~a~~ test function $v \in V$ and integrate

$$(\nabla u, \nabla v) - \int_\Gamma \partial_n u \, v \, ds = (f, v)$$

$$(\nabla u, \nabla v) - \int_{\Gamma_1} \cancel{\partial_n u \, v}^{\,0} \, ds - \int_{\Gamma_2} \partial_n u \, v \, ds = (f, v)$$

$$V = \left\{ v : v = 0 \text{ on } \Gamma_1 \text{ and } \int_\Omega (|\nabla v|^2 + v^2) \, dx < \infty \right\}$$

$$\partial_n u = -\gamma u + g$$

$$\Rightarrow (\nabla u, \nabla v) + \int_{\Gamma_2} \gamma u v \, ds - \int_{\Gamma_2} g v \, ds = (f, v)$$

<u>Variational formulation</u>: Find $u \in V$ s.t.

$$(V) \quad (\nabla u, \nabla v) + \int_{\Gamma_2} \gamma u v \, ds - \int_{\Gamma_2} g v \, ds = (f, v), \; \forall v \in V$$

* Robin and Neumann are imposed <u>weakly</u>

* Dirichlet bc. typically enforced <u>strongly</u> by the choice of functional space

From (V) we have:

$$\underbrace{(-\Delta u - f, v)}_{0} + \int_{\Gamma_2} (\partial_n u + \gamma(u - g)) v \, ds = 0$$

$$(R) \Rightarrow \underline{\int_{\Gamma_2} (\partial_n u + \gamma(u - g)) v \, ds = 0}$$

Weak implementation of Neumann / Dirichlet bc using Robin bc. From (R)

$$\partial_n u + \gamma(u - u_d) = g \quad \text{on } \Gamma_2$$

$$\gamma = 0 \Rightarrow \text{Neumann } \partial_n u = g \quad \text{on } \Gamma$$

$$\gamma = \infty \Rightarrow \text{Dirichlet } u = u_d \quad \text{on } \Gamma$$

* Adaptivity and error control

In lecture 4 we will prove the following a posteriori error estimate

$$\|\nabla u - \nabla U\|_{L_2} \leq C_i \, \|h R(U)\|$$

with the residual $R(U) = |f + \Delta U|$

$C_i \approx 0.1$ and $h = h(x)$ is the mesh function

For p.w. linear basis V_h: $\Delta U = 0$.

How can we approximate ΔU for $U \in V_h$?

Discrete Laplacian Δ_h

For a given W let $\Delta_h W$ be the unique function in V_h s.t.

$$-(\Delta_h W, v) = (\nabla W, \nabla v) \quad \forall v \in V_h$$

$$\Rightarrow \Delta_h U \in V_h \Rightarrow \Delta_h U = \sum_{j=1}^{M} \xi_j \phi_j$$

$$U \in V_h \Rightarrow U = \sum_{j=1}^{M} \eta_j \phi_j$$

Choose $v = \phi_i \Rightarrow$

$$-\underbrace{\sum_{j=1}^{M} \xi_j (\phi_j, \phi_i)}_{M} = \sum_{j=1}^{M} \eta_j (\nabla \phi_j, \nabla \phi_i)$$

$\Rightarrow$ $-M\xi = A\eta$, M - is the mass matrix.

Ones ξ is computed $\Rightarrow$ $\Delta_h U = \sum_{j=1}^{M} \xi_j \phi_j$

$\Rightarrow$ $R(U) = f + \Delta U \approx f + \Delta_h U$

To minimize the error $\|\nabla u - \nabla U\|$ we want to minimize $\|h R(U)\|$ so that we choose $h(x)$ small where $R(U)$ is large.

* Simple adaptive algorithm

Start with initial (coarse) mesh $\mathcal{T}_h^0$. Set $i=1$

(1) Compute solution $U \in V_h$ by FEM

(2) Compute $R(U) = f + \Delta U \approx f + \Delta_h U$

(3) Mark 50% (or) of the elements for refinement which have the largest residual $R(U)$

(4) Refine the mesh $\mathcal{T}_h^{i-1} \Rightarrow \mathcal{T}_h^{i}$

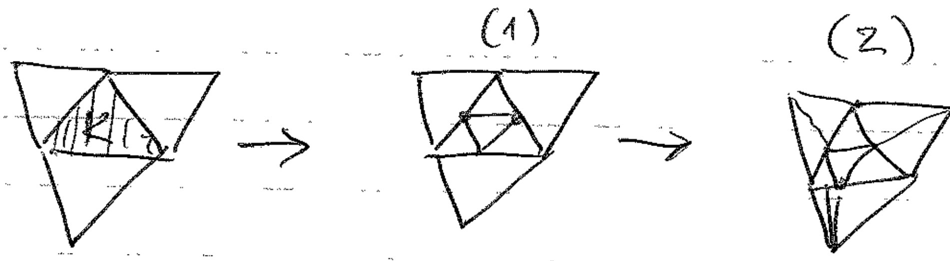
(5) Set $i = i+1$ then go to (1)

* Red-green mesh refinement

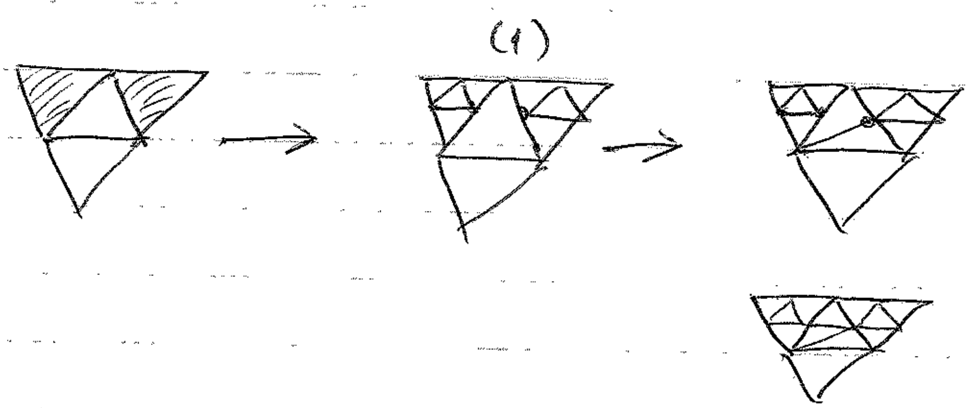
(1) Loop over all ~~cells~~ marked cells : Insert new nodes at edge midpoints and connect them by new edges $\Rightarrow$ 4 new cells

(2) Loop over all hanging nodes : connect each node with the node opposite in each cell.

Ex :



ex :



Futher reading :

CDE 15.1, 15.3, 15.4 (+ Robin bc).

Lecture 4

* Interpolation
* Error estimation
* High order FEM

Polynomial approximation

$\mathcal{P}^p(a,b)$ - the set of polynomial $p(x) = \sum_{i=0}^{p} c_i x^i$
of degree at most q on (a,b)
$c_i \in \mathbb{R}$ - coefficient of the polynom.

* A basis for $\mathcal{P}^q(a,b)$ consist of special set of polyn.
$\{1, x, x^2, \dots, x^q\}$

This set is linearly indimendend since

$$a_0 + a_1 x + a_2 x^2 + \dots + a_q x^q = 0 \quad \text{for all } x \in (a,b)$$

$$\Rightarrow a_0 = a_1 = \dots = a_q = 0$$

* Another basis for $\mathcal{P}^q(a,b)$ is Lagrange basis
$\{\lambda_i\}_{i=0}^{q}$; $q+1$ points $\xi_0 < \xi_1 < \dots < \xi_q$ in (a,b)

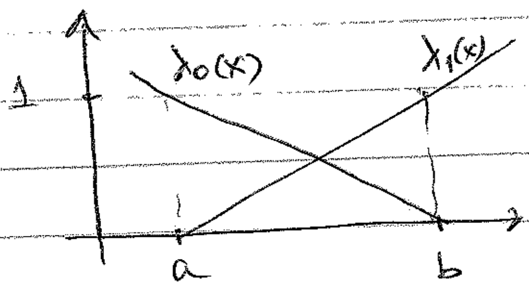
$$\lambda_i(\xi_j) = \begin{cases} 1 & ; \ i \neq j \\ 0 & , \ i \neq j \end{cases}$$

$$\lambda_i(x) = \prod_{j \neq i} \frac{x - \xi_j}{\xi_i - \xi_j} = \frac{(x-\xi_0)(x-\xi_1)\dots(x-\xi_q)}{(\xi_i - \xi_0)(\xi_i - \xi_1)\dots(\xi_i - \xi_p)}$$

Any $p \in \mathcal{P}^q(a,b)$ can be written as

$$p(x) = \sum_{i=0}^{q} p_i \lambda_i(x) \quad \text{with } p_i = p(\xi_i)$$

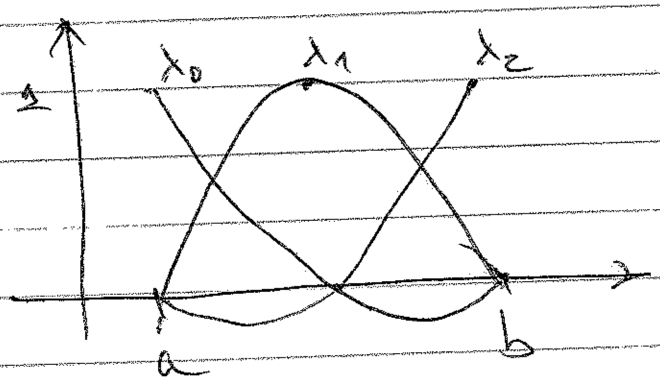
Ex: $q=1$: $\{\lambda_0(x), \lambda_1(x)\}$



$$\lambda_0(x) = \frac{x-\xi_1}{\xi_0-\xi_1}$$

$$\lambda_1(x) = \frac{x-\xi_0}{x_1-\xi_0}$$

Ex: $q=2$: $\{\lambda_0(x), \lambda_1(x), \lambda_2(x)\}$



$$\lambda_0(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$\lambda_1(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

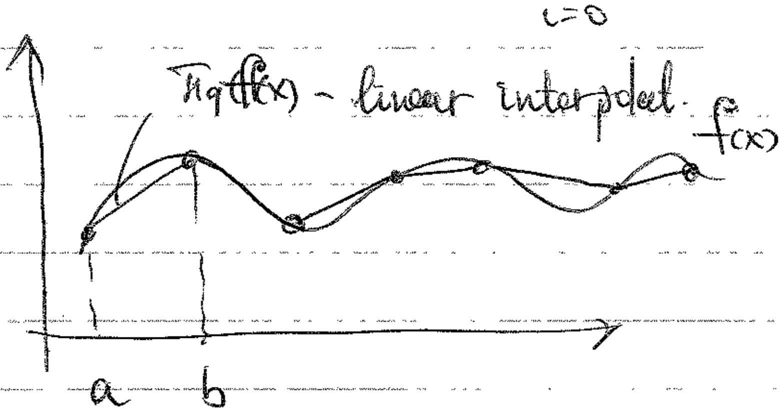
$$\lambda_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

* Polynomial interpolation

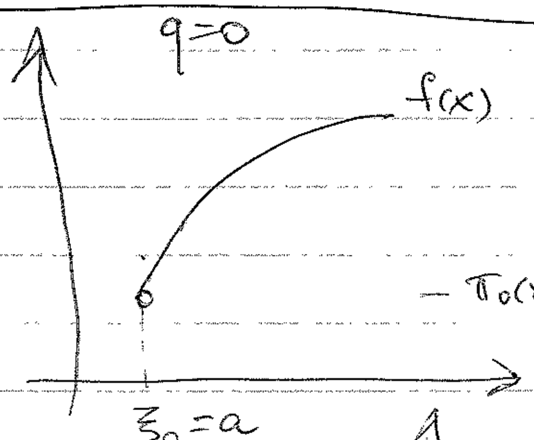
$\Pi_q f \in \mathbb{P}^q(a,b)$: $\Pi_q f(\xi_i) = f(\xi_i) \quad \forall i = 0,\dots,q$

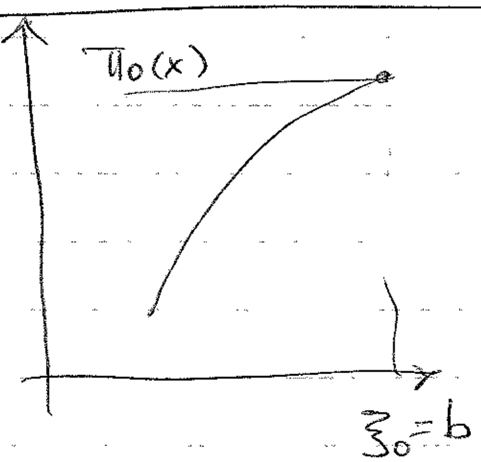
$f(x)$ - is a given function
$\Pi_q f$ - interpolant of f

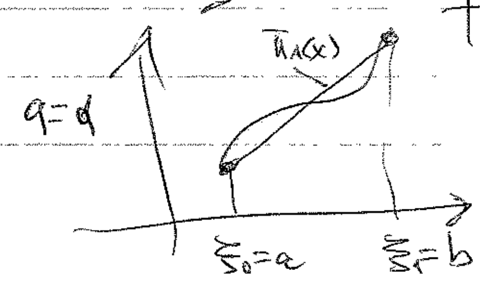
$$\Pi_q f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) + \dots + f(\xi_q)\lambda_q(x)$$
$$= \sum_{i=0}^{q} f(\xi_i)\lambda_i(x)$$



$$\Pi_1 f(x) = f(a)\,\frac{x-b}{a-b} + f(b)\,\frac{x-a}{b-a} \quad ; \quad \xi_0 = a, \ \xi_1 = b$$







* A pointwise estimate of the interpolation error

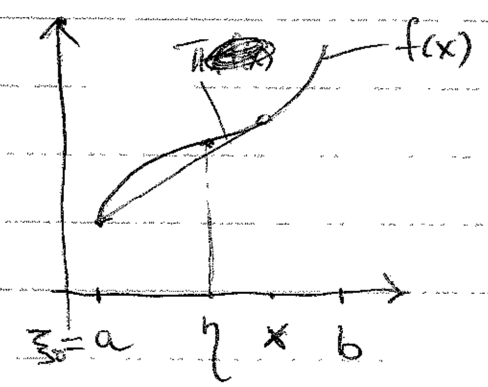
<u>Theorem 5.1.</u> Assume that f has $q+1$ continuous derivatives in (a,b), and let $\pi_q f \in \mathcal{P}^q(a,b)$ interpolate f at the points $a \le \xi_0 < \xi_1 < \dots < \xi_q \le b$. Then for $a \le x \le b$

$$(*) \quad |f(x) - \pi_q f(x)| \le \left| \frac{(x-\xi_0)\dots(x-\xi_q)}{(q+1)!} \right| \max_{[a,b]} |D^{q+1} f|$$

<u>Proof:</u> For the simplicity we proof for $q=0,1$.

a): $q=0$, then $(*)$ comes to

$$|f(x) - \pi_0 f(x)| \le |x-\xi_0| \max_{[a,b]} |f'|$$



The mean walue theorem:

$$f'(\eta) = \frac{f(x) - f(a)}{x-a}$$

$$\Rightarrow \quad f'(\eta) = \frac{f(x) - f(\xi_0)}{x - \xi_0}$$

$$f(x) - f(\xi_0) = (x-\xi_0) f'(\eta) \quad ; \quad \pi_0 f(x$$

$$|f(x) - f(\xi_0)| \le |x-\xi_0| \max_{[a,b]} |f'(x)| \,, \quad \forall x : a \le x \le b$$

~~a)~~ b) $q=1$. The error estimate ~~say~~ states that the error is propotional to $|f''|$

We want to prove that

$$(*) \rightarrow \left|f(x) - \pi_1 f(x)\right| \leq \frac{1}{2} |x-\xi_0||x-\xi_1| \max_{[a,b]} |f''|$$

Interpolation for $q=1$:

$$(1) \quad \pi_1 f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) = f(\xi_0)\frac{x-\xi_1}{\xi_0-\xi_1} + f(\xi_1)\frac{x-\xi_0}{\xi_1-\xi_0}$$

fixing x in (ξ_0, ξ_1) and using Taylor's theorem

put

$$(2) \quad f(\xi_i) = f(x) + f'(x)(\xi_i - x) + \frac{1}{2} f''(\eta_i)(\xi_i - x)^2$$

where $i=0,1$; η_i lies between x and ξ_i and using identities

(check) $\lambda_0(x) + \lambda_1(x) \equiv 1$; $(\xi_0 - x)\lambda_0(x) + (\xi_1 - x)\lambda_1(x) \equiv 0$

we obtain error represintation

$$f(x) - \pi_1 f(x) = -\frac{1}{2}\left(f''(\eta_0)(\xi_0 - x)^2\lambda_0(x) + f''(\eta_1)(\xi_1 - x)^2\lambda_1(x)\right)$$

Use formulas for λ_1 and λ_2, for $\xi_0 < x \leq \xi_1$ (5)

$$|f(x) - \Pi_1 f(x)| = \left| -\frac{1}{2}\left(f''(\eta_0)(\xi_0 - x)^2 \lambda_0(x) + f''(\eta_1)(\xi_1 - x)^2 \lambda_1(x)\right)\right|$$

$$\leq \frac{1}{2}\left(\frac{|\xi_0 - x|^2 \, |x - \xi_1|}{|\xi_0 - \xi_1|} |f''(\eta_0)| + \frac{|\xi_1 - x|^2 |x - \xi_0|}{|\xi_1 - \xi_0|} |f''(\eta_1)| \right)$$

$$\leq \frac{1}{2}\left(\frac{|\xi_0 - x|^2 \, |x - \xi_1|}{|\xi_0 - \xi_1|} + \frac{|\xi_1 - x|^2 |x - \xi_0|}{|\xi_1 - \xi_0|} \right) \max_{[a,b]} |f''|$$

$$= \frac{1}{2}\left(\frac{(x - \xi_0)^2 (\xi_1 - x)}{\xi_1 - \xi_0} + \frac{(\xi_1 - x)^2 (x - \xi_0)}{\xi_1 - \xi_0} \right) \max_{[a,b]} |f''|$$

$$= \frac{1}{2} |x - \xi_0| \, |x - \xi_1| \max_{[a,b]} |f''|$$

Theorem 5.2. For $\xi_0 \le x \le \xi_1$

$$\left| f'(x) - (\pi_1 f)'(x) \right| \le \frac{(x-\xi_0)^2 + (x-\xi_1)^2}{2(\xi_1 - \xi_0)} \max_{[a,b]} |f''|$$

Proof. Differentiating (1) ~~for x~~, with respect to x and using (2) with the identities

$$\lambda_0'(x) + \lambda_1'(x) \equiv 0, \quad (\xi_0 - x)\lambda_0'(x) + (\xi_1 - x)\lambda_1'(x) \equiv 1$$

we get the error representation formula

$$f'(x) - (\pi_1 f)'(x) = -\frac{1}{2}\Big(f''(\eta_0)(\xi_0 - x)^2 \lambda_0'(x) + f''(\eta_1)(\xi_1 - x)^2 \lambda_1'(x) \Big)$$

Taking absolute values and using

$$\left| \lambda_i'(x) \right| = \frac{1}{|\xi_1 - \xi_0|},$$

proves the theorem.

Maximum norm: $\|f\|_{L_\infty(a,b)} = \max_{[a,b]} |f(x)|$

L_1-norm: $\|f\|_{L_1(a,b)} = \int_a^b |f(x)|\, dx$

L_2-norm: $\|f\|_{L_2(a,b)} = \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}}$

Theorems 5.1 and 5.2 $\Rightarrow$

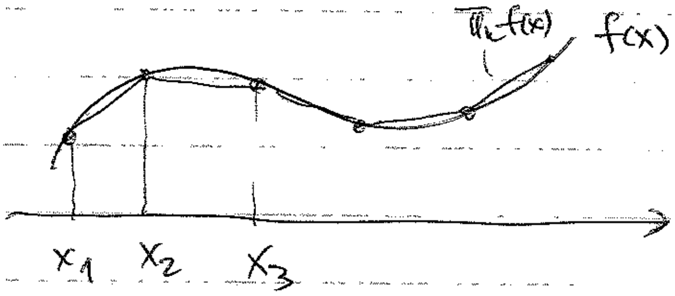
$$\|f - \pi_1 f\|_{L_\infty(a,b)} \le \frac{1}{8} (b-a)^2 \, \|f''\|_{L_\infty(a,b)}$$

$$\|f' - (\pi_1 f)'\|_{L_\infty(a,b)} \le \frac{1}{2} (b-a) \, \|f''\|_{L_\infty(a,b)}$$

here $\frac{1}{8}, \frac{1}{2}$ are the interpolations constants

Piecewise linear approximation on $\mathcal{T}_h = \{I_i\}$

$$V_h^{(1)} = \{ v \text{ contin on } I_i,\ v|_{I_i} \in \mathcal{P}^1(I_i),\ i = 1, 2, \dots, m+1 \}$$



$$\pi_h f(x) = f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \ ; \ x_{i-1} \le x \le x_i$$

for $i = 1, 2, \dots, m+1$

We already discussed:

$$U \in V_h^{(1)} \Rightarrow U = \sum_{i=0}^{m+1} U(x_i)\, \phi_i(x), \quad \phi_i(x) - \text{hat f.}$$

Then, there is:

Theorem 5.4. For $p = 1, 2$ and ∞, $\exists\, C_i$ - const s.t.

$$\| f - \pi_h f \|_{L_p(a,b)} \le C_i \| h^2 f'' \|_{L_p(a,b)}$$

$$\| f - \pi_h f \|_{L_p(a,b)} \le C_i \| h f' \|_{L_p(a,b)}$$

$$\| f' - (\pi_h f)' \|_{L_p(a,b)} \le C_i \| h f'' \|_{L_p(a,b)}$$

Lecture 5

* A priori error estimation
* A posteriori error estimation
* duality

* A priori error estimation

Assume 1D Poisson equation:

$$\begin{cases} -u'' = f \,, & a \le x \le b \\ u(a) = u(b) = 0 \end{cases}$$

(Wf) Find $u \in V$ s.t.

$$(W) \quad \int_a^b u'v'\,dx = \int_a^b fv\,dx \,, \quad \forall v \in V$$

$$V = \left\{ v : \int_a^b (|v'|^2 + |v|^2)\,dx < \infty \,;\ v(0) = v(1) = 0 \right\}$$

(GA): Find $U \in V_h$ s.t.

$$(G) \quad \int_a^b U'v'\,dx = \int_a^b fv\,dx \,, \quad \forall v \in V_h$$

$$V_h = \{ v : \text{cont. p.w. on subin } I_i \,;\ v(0) = v(1) = 0 \}$$

$$V_h \subset V \Rightarrow (W) - (G) : \int_a^b (u - U)' v'\,dx = 0 \,, \ \forall v \in V_h$$

$$\|(u-U)'\|^2_{L_2} = \int_a^b (u-U)'(u-U)'\,dx \cdot \{\text{add} - GO\}$$

$$= \int_a^b (u-U)'(u-U)'\,dx - \int_a^b (u-U)'(v-U)'\,dx$$

$$= \int_a^b (u-U)'(u-U-v+U)'\,dx = \int_a^b (u-U)'(u-v)'\,dx$$

$$\leq \|(u-U)'\|_{L_2}\,\|(u-v)'\|_{L_2}$$

$$\Rightarrow \quad \|(u-U)'\|_{L_2} \leq \|(u-v)'\|_{L_2}, \quad \forall v \in V_h$$

FEM solution $U \in V_h$ is optimal in energy norm!

Now choose $v = \pi_h u \in V_h$

$$\Rightarrow \quad \|(u-U)'\|_{L_2} \leq \|(u-\pi_h u)'\|_{L_2}$$

The interpolation error estimation gives that

$$\|(u-\pi u)'\|_{L_2} \leq C_i \|h\,u''\|_{L_2}$$

$$\Rightarrow \quad \underbrace{\|u-U\|_E = \|u'-U'\|_{L_2} \leq C_i \|h\,u''\|_{L_2}}_{\text{a priori error estimate}}$$

A posteriori error estimation

$$e = u - U \Rightarrow \|e'\|^2_{L_2} = \int_a^b e'e'\,dx$$

$$= \int_a^b e'(u-U)'dx = \left|\int_a^b \underbrace{e'u'}_{fe}\,dx - \int_a^b e'U'dx\right|$$

We use again the GO and $v = \pi_h e \in V_h$

$$\Rightarrow \|e'\|^2_{L_2} = \left|-\int_a^b f(e-\pi_h e)dx - \int_a^b U'(e-\pi_h e)'dx\right|$$

$$= \left|-\int_a^b f(e-\pi_h e)\,dx + \int_a^b U''(e-\pi e)\,dx\right|$$

$$= \left|\int_a^b R(U)(e-\pi_h e)dx\right| \leq \left\|\int_a^b R(U)\right\|_{L_2} \|e-\pi e\|_{L_2}$$

$$\leq \|R(U)\|_{L_2}\,\|h e'\|_{L_2} = h\,\|R(U)\|_{L_2}\,\cancel{\|e'\|_{L_2}}$$

$$\Rightarrow \|e'\|_{L_2} \leq h\,\|R(U)\|_{L_2}$$ a-posteriori error estimates

* Duality

Notation: let $x = [x_1, x_2, \dots, x_n]^T$

$$\|x\|^2 = (x, x) = x_1^2 + x_2^2 + \dots + x_n^2 ;$$

for $\phi(x)$: $\|\phi(x)\|_{L_2} = \left(\phi(x), \phi(x)\right) = \int_a^b |\phi(x)|^2 dx$

$x \in [a, b]$

To get a general feeling, let estimate a numerical error of a linear $n \times n$ system

$$(*) \qquad A\xi = b ;$$

We define $R = A\bar{\xi} - b$ - residual

$e = \xi - \bar{\xi}$ - error

$\bar{\xi}$ - approximate solution of $(*)$

ξ - exat solution of $(*)$

We start by posing the dual problem

$$(**) \qquad A^T \eta = e$$

Then; $\|e\|^2 = (e, e) = (e, A^T\eta) = (Ae, \eta) =$

$$= (A\xi - A\bar{\xi}, \eta) = (b - A\bar{\xi}, \eta) = (-R, \eta)$$

Suppose it is possible to estimate η

(***) $\|\eta\| \leq S\|e\|$

where S is a stability factor. Then by Cauchy-Scwarts inequality

$$\|e\|^2 = (-R, \eta) \leq \|R\|\,\|\eta\| \leq S\|R\|\,\|e\|$$

$$\Rightarrow \|e\| \leq S\|R\|$$

This is an a posteriori error estimate for the error e in terms of the residual R and the stability factor S.

We guaranty (***) by defining the stab. fac.

$$S = \max_{\theta \in \mathbb{R}^n, \theta \neq 0} \frac{|\xi|}{|\theta|}$$

where ξ solves $A^T\xi = \theta$

* Definition

Assume u and v sufficiently smooth with compact support in Ω.

Def. Let L be a differential operator or linear transformation. The formal adjoint L^* is the diff. operator that satisfies

$$(Lu, v) = (u, L^* v); \quad \left(\int_\Omega Lu \cdot v \, dx = \int_\Omega v \cdot L^* u \, dx \right)$$

Example 1. $L \equiv A$, $n \times n$ matrix: for vectors x, y

$$(Lx, y) = (Ax, y) = (x, A^T y) = (u, L^* v)$$

$$\Rightarrow \quad L^* = A^T$$

Example 2. $L = \frac{\partial}{\partial x}$; $u, v \in \Omega$, compact supp.

$$(Lu, v) = \left(\frac{\partial}{\partial x} u, v\right) = \left(u, -\frac{\partial}{\partial x} v\right) = (u, L^* v)$$

$$\Rightarrow \quad L^* = -\frac{\partial}{\partial x}$$

Example 3. $L = \Delta$; $u, v \in \Omega$, compact supp.

$$(Lu, v) = (\Delta u, v) = (\nabla u, -\nabla v) = (u, \Delta v) = (u, L^* v)$$

$$\Rightarrow \quad L^* = \Delta \quad \text{— self adjoint.}$$

* An a posteriori error estimation for a two-point BVP

$$\begin{cases} -(au')' + cu = f, & \text{in } (0,1) \\ u(0)=0, \; u(1)=0 \end{cases}$$

with $a(x)>0$, $c(x)\geq 0$.

(Wf) Find $u \in V$ s.t.

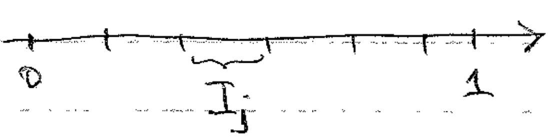
$$\int_0^1 (au'v' + cuv)\,dx = \int_0^1 fv\,dx, \quad \forall v \in V$$

$$V = \left\{ v: \int_0^1 (|v'|^2 + v^2)\,dx < \infty, \; v(0)=v(1)=0 \right\}$$

(GA): Find $U \in V_h$ s.t.

$$\int_0^1 (aU'v' + cUv)\,dx = \int_0^1 fv\,dx, \quad \forall v \in V_h$$

$$V_h = \{ v: v \text{ cont. p.w. linear on } I_j, \; v(0)=v(1)=0 \}$$



$I_j = (x_{j-1}, x_j)$

We want to estimate the L_2 norm of the error $e = u - U$. To do this we introduce the <u>dual problem</u>:

$$\begin{cases} -(a\varphi')' + c\varphi = e & \text{in } (0,1) \\ \varphi(0) = \varphi(1) = 0 \end{cases}$$

Error estimation:

$$\|e\|^2 = \int_0^1 e\left(-(a\varphi')' + c\varphi\right) dx = \int_0^1 (ae'\varphi' + ce\varphi) dx$$

$$= \int_0^1 (au'\varphi' + cu\varphi) dx - \int_0^1 (aU'\varphi' + cU\varphi) dx$$

$$= \int_0^1 \left(-(au')' + cu\right)\varphi \, dx - \int_0^1 (aU'\varphi' + cU\varphi) dx$$

$$= \int_0^1 f\varphi \, dx - \int_0^1 (aU'\varphi' + cU\varphi) dx$$

$$= \int_0^1 f(\varphi - \Pi_h\varphi) dx - \sum_{j=1}^{M+1} \int_{I_j} aU'(\varphi - \Pi_h\varphi)' + cU(\varphi - \Pi_h\varphi)) dx$$

$I_j = (x_j - x_{j-1})$ - subint.; integration by parts

$$= \int_0^1 f(\varphi - \Pi_h\varphi) dx - \sum_{j=1}^{M+1} \int_{I_j} \left(-(aU')' + cU\right)(\varphi - \Pi_h\varphi) dx$$

All the boundary terms dissappear since $(\varphi - \Pi_h\varphi)(x_j) = 0$

$$= \int_0^1 R(U)\,(\varphi - \Pi_h\varphi) dx$$

$$= \int_0^1 h^2 R(U) \frac{1}{h^2} (\varphi - \Pi_h\varphi) dx \leq$$

$$\leq \|h^2 R(U)\| \, \|h^{-2}(\varphi - \Pi_h\varphi)\| \leq \|h^2 R(U)\| \, C_i \|\varphi''\|$$

$R(U) = f + (aU')' - cU$ - residual

We define the stability factor as

$$S = \max_{\xi \in L_2(I)} \frac{\|\varphi''\|_{L_2}}{\|\xi\|_{L_2}}$$

$$\Rightarrow \|e\|^2_{L_2} \le C_i \|h^2 R(U)\| \cdot \frac{\|\varphi''\|}{\|e\|} \cdot \|e\|$$

$$\|e\| \le C_i \|h^2 R(U)\| \frac{\|\varphi''\|}{\|e\|}$$

$$\le S C_i \|h^2 R(U)\|.$$

* Adaptive error control

Given TOL > 0, a tolerance, choose $h = h(x)$ such that $S C_i \|h^2 R(U)\| <$ TOL.

Adaptive algorithm:

1. Choose initial ~~mesh~~ coarse mesh $\mathcal{T}_h^{(0)}$
2. Compute FEM solution $U \in V_h$
3. Compute residual $R(U)$
4. Solve dual problem for φ
5. Compute $S C_i \|h^2 R(U)\|$, if it is smaller than TOL, stop, otherwise
6. Refine the mesh where $\int_K S C_i (h_K^2 R(U))^2 dx$ is large
7. Go to 2.