Lecture 6

* Abstract problem, Lax-Millgram

Abstract framework

(i) a Hilber space V where we look for the solution, with norm $\|\cdot\|_V$ and sc. pr. $(\cdot,\cdot)_V$
(ii) a bilinear form $a: V\times V \to \mathbb{R}$, that is determined by the underlying DE.
(iii) a linear form $L: V \to \mathbb{R}$ that is determined by the data

We will formulate our DE using a bilinear form and a linear form, and search for a solution in the Hilbert space V.
A <u>bilinear form</u> $a(\cdot,\cdot)$ is a function taking values in $V\times V$ into $\mathbb{R}$. That is, $a(v,w) \in \mathbb{R}$ for all $v, w \in V$ such that $a(v,w)$ is linear in each argument.
$a(\alpha_1 v_1 + \alpha_2 v_2, w_1) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_2, w_1)$ and
$a(v_1, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 a(v_1, w_1) + \alpha_2 a(v_1, w_2)$
for all $\alpha_i \in \mathbb{R}$, $v_i, w_i \in V$.

A <u>linear form</u> $L(\cdot)$ is a function on V s.t. $L(v) \in \mathbb{R}$ $\forall\, v \in V$ and linear in v:

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$$

The <u>abstract problem</u> is : Find $u \in V$ s.t.

$$(*) \quad a(u,v) = L(v) \quad \forall v \in V$$

<u>Problem</u> : Do such solutions $u \in V$ exist?
<u>Problem</u> : If so: is such a solution unique?

"Existence" and "Uniqueness".

Depending $a(\cdot,\cdot)$, $L(\cdot)$, and V, we may prove existence and uniqueness of solutions.

- Assume that $a(\cdot,\cdot)$ is <u>V-elliptic</u> or <u>coercive</u> that means, $\exists K_1 > 0$ s.t.

$$a(v,v) \geq K_1 \|v\|_V^2 \quad , \forall v \in V$$

- $a(\cdot,\cdot)$ is <u>continuous</u>: $\exists K_2$ s.t.

$$|a(v,w)| \leq K_2 \|v\|_V \|w\|_V \, , \, \forall v, v \in V$$

- $L(\cdot)$ is <u>continuous</u>. $\exists K_3$ s.t.

$$|L(v)| \leq K_3 \|v\|_V \, , \, \forall v \in V$$

Lax - Milgram theorem

Suppose $a(\cdot,\cdot)$ is a continuous, V-elliptic bilinear form on the Hilbert space V, and $L(\cdot)$ is a linear form on V. Then there is a unique element $u \in V$ satisfying (*) and

$$\|u\|_V \le \frac{K_3}{K_1}$$

(Read proof from the book).

L is continuous since by linearity

$$|L(v) - L(w)| = |L(v-w)| \le K_3 \|v-w\|_V$$

so that $L(v) \to L(w)$ if $\|v-w\|_V \to 0$

($v \to w$ in V)

Similar with $a(\cdot,\cdot)$

- $\|v\|_a = \sqrt{a(v,v)}$ — energy norm

$\|v\|_a \ge 0$ since $\|v\|_a^2 = a(v,v) \ge K_1 \|v\|_V^2 \ge 0$

- $K_1 \|v\|_V^2 \le \|v\|_a^2 \le K_2 \|v\|_V^2$

$\Rightarrow$ $\|\cdot\|_a$ and $\|\cdot\|_V$ are equivalent norms.

if we take $K_1 = K_2 = 1 \Rightarrow \|\cdot\|_V = \|\cdot\|_a$

The abstract Galerkin method

Find $U \in V_h \subset V$ such that

(GA) $\quad a(U, v) = L(v)$ for all $v \in V_h$

Where $V_h \subset V$ is a finite dimentional space.

Galerkin orthogonality:

$$a(u-U, v) = 0 \quad \forall v \in V_h$$

A priori error estimates: Theorem 21.3

If u statisfies (WF) and U sat. (GA) then

$$\|u - U\|_V \leq \frac{K_2}{K_1} \|u - v\|_V.$$

if $\|\cdot\|_V = \|\cdot\|_a$, then

$$\|u - U\|_a \leq \|u - v\|_a.$$

~~which expresses~~ Galerkin solution is optimal in energy norm!)

Proof. The V-ellipticity and continuouty of a + (GA)

$$K_1 \|u-U\|_V^2 \leq a(u-U, u-U) = a(u-U, u-U) + a(u-U, U-v)$$

$$= a(u-U, u-v) \leq K_2 \|u-U\| \|u-v\|_V$$

The Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$

$$H^1(\Omega) = \left\{ v : \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty \right\}$$

$$(v,w)_{H^1(\Omega)} = \int_\Omega (\nabla v \cdot \nabla w + vw)\,dx$$

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega (|\nabla v|^2 + v^2)\,dx \right)^{\frac{1}{2}}$$

$H_0^1(\Omega) \subset H^1(\Omega)$, with the same norm and scalar product s.t.

$$H_0^1(\Omega) = \left\{ v : \{ v \in H^1(\Omega) \,;\, v = 0 \text{ on } \Gamma \right\}$$

Γ - is boundary of Ω

Th. 21.4

Poincare - Friddrichs inequality: There is a const C depending on Ω such that for all $v \in H^1(\Omega)$

$$\|v\|^2_{L_2(\Omega)} \le C\left(\|v\|^2_{L_2(\Gamma)} + \|\nabla v\|^2_{L_2(\Omega)} \right)$$

Theorem 21.5. If Ω is a bounded domain with boundary Γ, then there is a constant C s.t $\forall v \in H^1(\Omega)$

$$\|v\|_{L_2(\Gamma)} \le C \|v\|_{H^1(\Omega)}$$

Example A problem with Neumann bc's

Consider Poisson's equation with an absortion term together with Neumann bc's

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \quad , \ \Omega \subset \mathbb{R}^d \text{, bounded} \\ \partial_n u = 0 & \text{on } \Gamma \quad , \ \Gamma = \partial\Omega \end{cases}$$

(Wf) Find $u \in V = \{v : \int_\Omega (|\nabla u|^2 + v^2)\,dx < \infty\}$ s.t.

$$\int_\Omega (\nabla u \cdot \nabla v + uv)\,dx = \int_\Omega f v\,dx, \quad \forall v \in V$$

$$\Rightarrow \ a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx \ , \quad L(v) = \int_\Omega f v\,dx$$

* V– Hibert space? Yes, since

$$(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx \ ; \quad \|v\|_V = \left(\int_\Omega (|\nabla v|^2 + v^2)\,dx \right)^{\frac{1}{2}}$$

V – complete, (it comes up by completeness of $\mathbb{R}$) since it is a Sobolev space

*(i) $a(\cdot,\cdot)$ – V-elliptic?

$$a(v,v) = \int_\Omega (|\nabla v|^2 + v^2)\,dx = \|v\|_V^2 \ , \quad a(v,v) = \|v\|_V^2$$

with $k_1 = 1$

(ii) $a(\cdot,\cdot)$ - continuous?

$$|a(v,w)| = \left|\int_\Omega (\nabla v \cdot \nabla w + v \cdot w)\,dx\right| \leq \underbrace{\|\nabla v\|_{L_2}\|\nabla w\|_{L_2}}_{a} + \underbrace{\|v\|_{L_2}\|w\|_{L_2}}_{b}$$

$$\cancel{\leq \|\nabla v\|_{L_2}\|\nabla w\|_{L_2} + \left(\|\nabla v\|_{L_2}\|w\|_{L_2} + \|\nabla w\|_{L_2}\|v\|_{L_2}\right) + \|v\|_{L_2}\|w\|_{L_2}}$$

$$\cancel{= \left(\|\nabla v\|_{L_2} + \|v\|_{L_2}\right)\left(\|\nabla w\|_{L_2} + \|w\|_{L_2}\right) = \|v\|}$$

$$= a + b = \sqrt{(a+b)^2} = \sqrt{a^2+b^2+2ac} \leq$$

$$= \sqrt{\|\nabla v\|^2\|\nabla w\|^2 + \|v\|^2\|w\|^2 + 2\|\nabla v\|\|\nabla w\|\|w\|\|v\|} \leq$$

$$\leq \sqrt{\|\nabla v\|^2\|\nabla w\|^2 + \|v\|^2\|w\|^2 + \|\nabla v\|^2\|w\|^2 + \|\nabla w\|^2\|v\|^2}$$

$$= \sqrt{\left(\|\nabla v\|^2 + \|v\|^2\right)\left(\|\nabla w\|^2 + \|w\|^2\right)} = \|v\|_H \|w\|_V$$

$$\Rightarrow |a(v,w)| \leq \|v\|_V \|w\|_W \quad \text{with } k_2 = 1$$

(iii) $L(\cdot)$ - continuous?

$$|L(v)| = \left|\int_\Omega f v\,dx\right| \leq \|f\|_{L_2}\|v\|_{L_2} \overset{\text{Poincare}}{\leq} \|f\|_{L_2}\|v\|_{H^1}$$

$$= \|f\|_{L_2}\|v\|_V$$

L - continuous with $k_3 = \|f\|_{L_2(\Omega)}$

$\Rightarrow$ L-M applies (*) has a unique solution.

$$(Lu, \varphi) = (u, L^* \varphi)$$

Lecture 7

* Initial value problem (IVP)

$$\begin{cases} \dot{u}(x,t) + A(u(x,t)) = f(x,t) , & \forall x,t \in \Omega \times [0,T] \\ u(x,t) = w(x,t) & \forall x \in \partial\Omega \times [0,T] \\ u(x,0) = u_0(x) & \forall x \in \Omega \end{cases}$$

Timestepping: Discretization in time $[0,T]$

$t_0=0 \quad t_1 \quad t_2 \quad \cdots \quad t_m$

solution at time $t=t_m$ is given by solution and data at earlier steps. t_n, $n<m$

Stability: growth / decay of perturbations of a solution with time

Usually time and space discretizations introduces perturbations. In general the error accumulates (grow in time).

Parabolic problem: $(Av,v) \geq 0$, $(Av,w) = (v,Aw)$
$\forall v,w \in$

Parablic problems are dissipative: error do not accumilate in time!

Example Heat equation

$$(*)\begin{cases} \dot{u} + \Delta u = f, & (x,t) \in \Omega \times [0,T] \\ u = 0, & x \in \partial\Omega \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

$$(Av,v) = (-\Delta v, v) = (\nabla v, \nabla v) = \|\nabla v\|^2 \geq 0$$

$$(Av,w) = (-\Delta v, w) = (v, -\Delta w) = (v, Aw) \quad , \forall v, w$$

$\Rightarrow$ ~~The~~ Heat equation is parabolic!

* Stabity / Energy estimates

(*) We multiply (*) by $u(x)$ and integrate by time: and assuming that $f=0$

1) $$\int_\Omega (\dot{u} + \Delta u) u \, dx = \int_\Omega \dot{u} u \, dx - \int_\Omega \Delta u \, u \, dx$$

$$= \frac{1}{2} \int_\Omega \frac{d}{dt}(|u|^2) dx + \int_\Omega \nabla u \nabla u \, dx$$

$$= \frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx + \int_\Omega \nabla u \nabla u \, dx = \frac{1}{2} \frac{d}{dt} \|u\|^2_{L_2} + \|\nabla u\|^2_{L_2}$$

$$\frac{d}{dt}\int_\Omega |u|^2 dx = \int_\Omega \frac{d}{dt}(u_1^2 + u_2^2 + u_3^2) dx = \int_\Omega 2u_1\dot{u}_1 + 2u_2\dot{u}_2 + 2u_3\dot{u}_3 = 2(\dot{u}, u)$$

If we integrate in time we get

$$(\cancel{*}*) \quad \boxed{\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2 dt \leq \|u_0\|^2} \quad *$$

$$\|u(T)\|^2 + 2\int_0^1 (\nabla u(t), \nabla u(t))\, dt \leq \|u_0\|^2$$

2) Now multiply $(*)$ by $-t\Delta u$ and integrate

$$(\dot{u}, -t\Delta u) - (\Delta u, -t\Delta u) = 0$$

$$(\dot{u}, -t\Delta u) = t(\nabla\dot{u}, \nabla u) = \frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) - \frac{1}{2}\|\nabla u\|$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(t\|\nabla u\|^2\right) - \frac{1}{2}\|\nabla u\|^2 + t\|\Delta u\|^2 = 0$$

Integrat in time

$$\frac{T}{2}\|\nabla u(T)\|^2 + \int_0^T t\|\Delta u(t)\|^2 dt = \frac{1}{2}\int_0^T \|\nabla u\|^2 dt$$

$$\Rightarrow \int_0^T t\|\Delta u(t)\|^2 dt = \frac{1}{2}\left(\int_0^T \|\nabla u\|^2 dt - T\|\nabla u(T)\|^2\right)$$

$\boxed{\text{from } (\cancel{*}*) \Rightarrow}$ $$= \frac{1}{2}\left(\frac{1}{2}\|u_0\|^2 \underbrace{- \frac{1}{2}\|u(T)\|^2 - T\|\nabla u(T)\|^2}\right)$$

$$\leq \frac{1}{4}\|u_0\|^2$$

3) We now multiply $(*)$ to $t^2 \|\Delta^2 u(t)\|^2$ & integrate

$$\Rightarrow \boxed{\left(\mathring{u}, t^2 (\Delta^2 u)\right)} - \left(\Delta u, t^2 (\Delta^2 u)\right) = 0$$

a) $$\frac{1}{2}\frac{d}{dt}\left(t^2 \|\Delta u\|^2\right) = t\|\Delta u\|^2 + t^2(\Delta \mathring{u}, \Delta u) =$$

$$= t\|\Delta u\|^2 + \boxed{\left(\mathring{u}, t^2 \Delta^2 u\right)}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(t^2\|\Delta u\|^2\right) - t\|\Delta u\|^2 - \left(\Delta u, t^2 \Delta^2 u\right) = 0$$

$$\frac{1}{2}\frac{d}{dt}\left(t^2\|\Delta u\|^2\right) = t\|\Delta u\|^2 + \boxed{\left(\Delta u, t^2 \Delta^2 u\right)}$$

$$\left(\Delta u, t^2 \Delta^2 u\right) = -\left(\nabla(\Delta u), t^2 \nabla(\Delta u)\right) = \underbrace{-\left(t^2 \|\nabla(\Delta u)\|^2\right)}_{\leq 0}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(t^2\|\Delta u\|^2\right) \leq t\|\Delta u\|^2$$

Integrate in time:

$$\frac{T^2}{2}\|\Delta u(T)\|^2 \leq \int_0^T t\|\Delta u(t)\|^2 dt \overset{\text{from 2)}}{\leq} \frac{1}{4}\|u_0\|^2$$

$$\|\Delta u(T)\| \leq \frac{1}{\sqrt{2}\,T}\|u_0\|$$

We had that for solution u of Heat equation

1) $\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2 dt = \|u_0\|^2$

2) $\int_0^T t\,\|\Delta u\|^2 dt \leq \frac{1}{4}\|u_0\|^2$

3) $\|\Delta u(T)\| \leq \frac{1}{\sqrt{2}\,T}\|u_0\|^2$

1–3 are called $\boxed{\text{strong stability}}$ estimates, because all $\boxed{\text{derivatives}}$ of the equation and its $\boxed{\text{solution}}$ is bounded by the $\boxed{\text{intial data}}$

For example 3 implies that for $f=0$

$$\|\dot{u}(T)\| = \|\Delta u(T)\| \leq \frac{1}{\sqrt{2}\,T}\|u_0\|^2$$

$$\Rightarrow \|\dot{u}(T)\| \sim \frac{1}{T}, \quad \begin{array}{l} \to 0 \text{ if } T \to \infty \\ \to \infty \text{ if } T \to 0 \end{array}$$

$\Rightarrow$ The solution becomes smoother ~~if~~ as time passes

When $f \neq 0$: multiply (*) by u and integrate

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = |(f,u)|$$

* Cauchy-Scwartz inequality:

$$|(f,u)| \leq \|f\|\,\|u\|$$

* Variant of Cauchy inequality:

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2}b^2 \quad , \forall a,b \, , \varepsilon > 0$$

Proof: $a^2 - 2\varepsilon ab + \varepsilon^2 b^2 = (a-\varepsilon b)^2 \geq 0$

$$\Rightarrow \quad |(f,u)| \leq \|f\|\,\|u\| \leq \frac{1}{2c}\|f\|^2 + \frac{c}{2}\|u\|^2 , \; c>0$$

* Poincare-Fredrich inequality: $\exists\, c > 0$ s.t

$$\|\nabla v\|^2 \geq c\|v\|^2 \quad \forall v \in H_0^1(\Omega)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = |(f,u)|$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \underline{c\|u\|^2} \leq |(f,u)| \leq \frac{1}{2c}\|f\|^2 + \underline{\frac{c}{2}\|u\|^2}$$

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2 \leq \frac{1}{c}\|f\|^2$$

since $c\|u\|^2 \geq 0$, we have that

$$\frac{d}{dt}\|u\|^2 \leq \frac{1}{c}\|f\|^2$$ integrat in time:

$$\|u(T)\|^2 - \|u(0)\|^2 \leq \frac{1}{c}\int_0^T \|f\|^2 dt$$

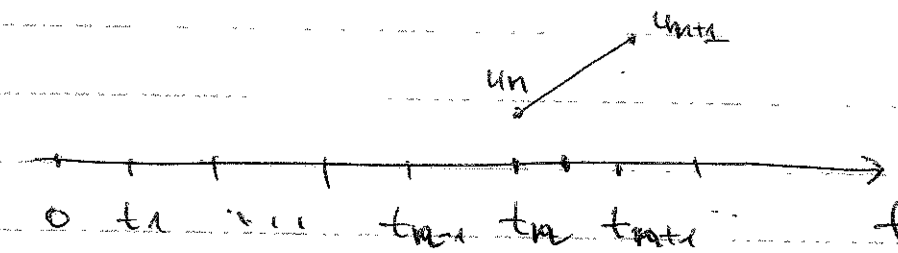
$$\Rightarrow \|u(T)\|^2 \leq \|u(0)\|^2 + \frac{1}{c}\int_0^T \|f\|^2 dt$$

* Time discrelization with the θ-method

Multiply the Heat equation with test func v and integrate in space we get

(W.f) Find $u \in V$ s.t.

$$(\dot{u}, v) + a(u,v) = 0 \quad \text{with } a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall v \in V$$



on interval $t_n \le t \le t_{n+1}$ we make a linear approximation

$$u(t) \approx \frac{t - t_n}{k_n} u^{n+1} + \left(1 - \frac{t - t_n}{k_n}\right) u^n$$

with $u_n = u(t_n)$, $k_n = t_{n+1} - t_n$; Let $\theta(t) = \frac{t - t_n}{k_n}$

$$u(t) \approx \theta u^{n+1} + (1-\theta) u^n = u^{n+\theta}$$

$$\Rightarrow \left(\frac{u^{n+1} - u^n}{k_n}, v\right) + a\left(u^{n+\theta}, v\right) = 0 \quad \forall v \in V_h$$

$$(f^{n+\theta}, v)$$

When $\begin{cases} \theta = 0 \Rightarrow \text{Forward Euler method} \\ \theta = 1 \Rightarrow \text{Backward Euler} \\ \theta = \frac{1}{2} \Rightarrow \text{Crank–Nicolson} \end{cases}$

Example

(WF). Find $u \in V = \{v : \int_\Omega (|\nabla v|^2 + v^2)\,dx < \infty,\ u = 0 \text{ on } \partial\Omega\}$

s.t.

$$(\dot{u}, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in V$$

Where $(u,v) = \int_\Omega uv\,dx$

(GA). Find $U \in V_h = \{v : v \in V,\ v\text{- p.w. cont on cells } K_i;\ \Omega = \bigcup_i^M K_i\}$

$$(\dot{U}, v) + (\nabla U, \nabla v) = (f, v), \quad \forall v \in V_h$$

if $U \in V_h \Rightarrow \quad U = \sum_{j=1}^{N} \varphi_j(x,y)\, \xi_j(t)$

$$\Rightarrow \left(\sum_{j=1}^{N} \dot{\xi}_j(t)\varphi_j, \varphi_i\right) + \left(\sum_{j=1}^{N} \xi_j(t)\nabla\varphi_j, \nabla\varphi_i\right) = (f, \varphi_i), \quad i = 1, \dots, N$$

$$\Rightarrow \underbrace{\sum_{j=1}^{N} (\varphi_j, \varphi_i)}_{M} \dot{\xi}_j(t) + \underbrace{\sum_{j=1}^{N} (\nabla\varphi_j, \nabla\varphi_i)}_{A} \xi_j(t) = (f, \varphi_i) \quad \underbrace{i = 1, \dots, N}_{F}$$

$$\Rightarrow M\dot{\xi} + A\xi = F$$

We discretize with forward ~~Euler θ=0~~

$$\Rightarrow M \frac{\xi^{n+1}-\xi^{n}}{k_n} + S\left(\theta \xi^{n+1} + (1-\theta)\xi^{n}\right) = \theta f^{n+1} + (1-\theta) f^{n}$$

1) $\theta = 0$; Forward Euler

$$M \frac{\xi^{n+1}-\xi^{n}}{k_n} + S\xi^{n} = f^{n}$$

$$\Rightarrow M\xi^{n+1} = M\xi^{n} - k_n\left(S\xi^{n} - f^{n}\right)$$

2) $\theta = 1$; backward Euler

$$M \frac{\xi^{n+1}-\xi^{n}}{k_n} + S\xi^{n+1} = f^{n+1}$$

$$\Rightarrow M\xi^{n+1} + k_n\left(S\xi^{n+1} - f^{n+1}\right) = M\xi^{n}$$

3) $\theta = \frac{1}{2}$: Crank-Nicolson

$$M \frac{\xi^{n+1}-\xi^{n}}{k_n} + S\left(\frac{1}{2}\left(\xi^{n+1}+\xi^{n}\right)\right) = \frac{1}{2}\left(f^{n+1}+f^{n}\right)$$

$$\Rightarrow M\xi^{n+1} + \frac{k_n}{2}\left(S\xi^{n+1} - f^{n+1}\right) = M\xi^{n} - \frac{k_n}{2}\left(S\xi^{n} - f^{n}\right)$$

* Wave equations

$$(1)\begin{cases} \ddot{u} - \Delta u = f & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \Gamma \times (0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \\ \dot{u}(x,0) = \mathring{u}_0(x) & \text{in } \Omega \end{cases}$$

- Multiply (1) by $\dot{u}$ and integrate over Ω
  Set $f=0$ $\Rightarrow$

- $$(\ddot{u} - \Delta u, \dot{u}) = (\ddot{u}, \dot{u}) - (\Delta u, \dot{u}) = \underbrace{(\ddot{u}, \dot{u}) + (\nabla u, \nabla \dot{u})}_{=0}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(\|\dot{u}\|^2 + \|\nabla u\|^2\right) = 0$$

Energy does not change in time : conservative!

- As we did for Heat equation :

- (wf). Find $u \in V$ s.t.

$$(\ddot{u}, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in V$$

(GA) Find $U \in V_h \subset V$ s.t.

$$(\ddot{U}, v) + (\nabla U, \nabla v) = (f, v), \quad \forall v \in V_h$$

Now insert $U=\sum_{j=1}^{N} \xi_j(t)\,\varphi(x,y)$ and

chose $v=\varphi_i(x,t) \Rightarrow$

$$M\frac{d^2\xi}{dt^2}+S\xi=f$$

$$M=\int_\Omega \varphi_i\varphi_j\,dx\ ,\quad S=\int_\Omega \nabla\varphi_i\nabla\varphi_j\,dx$$

Let $\eta=\frac{d\xi}{dt} \Rightarrow \begin{cases} \eta=\frac{d\xi}{dt} \\ M\frac{d\eta}{dt}+S\xi=f \end{cases}$

Discretize in time with θ-method:

$$\begin{cases} \frac{\xi^{n+1}-\xi^{n}}{k_n}=\theta\eta^{n+1}+(1-\theta)\eta^{n} \\ M\frac{\eta^{n+1}-\eta^{n}}{k_n}+S\left(\theta\xi^{n+1}+(1-\theta)\xi^{n}\right)=\theta f^{n+1}+(1-\theta)f^{n} \end{cases}$$

Lecture 8

* Convection-diffusion-reaction equation
* Space-time FEM
* Stabilization

$Q = \Omega \times I$, $\Omega \subset \mathbb{R}^2$, $I \in [0, T)$, $\Gamma = \partial\Omega$

$$
\begin{cases}
\dot{u} + \nabla\cdot(\beta u) + \alpha u - \nabla\cdot(\varepsilon \nabla u) = f & \text{in } Q \\
u = g_- & \text{on } (\Gamma \times I)_- \\
u = g_+ \text{ or } \varepsilon \partial_n u = g_+ & \text{on } (\Gamma \times I)_+ \\
u(\cdot, 0) = u_0 & \text{in } \Omega
\end{cases}
$$

where u - temperature

$\beta = (\beta_1, \beta_2)$ - convection field

α, ~~ε~~ absorrption coef

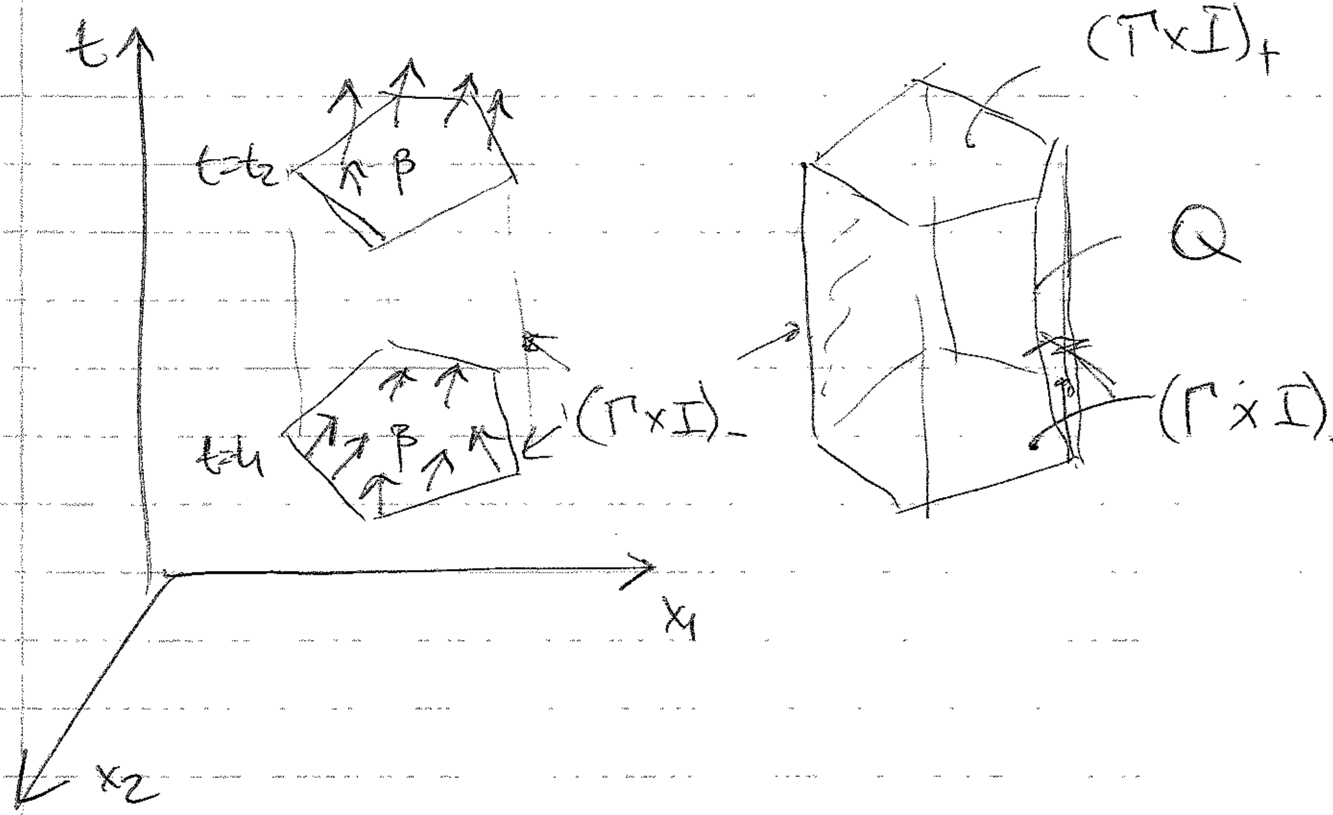
$\varepsilon > 0$ - diffusion coef

$f(x,t)$, u_0, g, - are given data

$(\Gamma \times I)_- = \{(x,t) \in \Gamma \times I : \beta(x,t)\cdot n(x) < 0\}$ inflow

$(\Gamma \times I)_+ = \{(x,t) \in \Gamma \times I : \beta(x,t)\cdot n(x) \geq 0\}$ outflow

$n(x)$ - ~~norm~~ outward normal to Γ at x.



We may use divergence form non-divergence form

$$\nabla\cdot(\beta u) = \beta\cdot\nabla u + (\nabla\cdot\beta)u$$

$$\Rightarrow \quad \nabla\cdot(\beta u) + \alpha u = \left(\beta\cdot\nabla u + (\nabla\cdot\beta)u\right) + \alpha u$$

$$= \beta\cdot\nabla u + \underbrace{(\alpha + \nabla\cdot\beta)}_{\tilde{\alpha}}u$$

Is the temperature conserved?

$$\frac{d}{dt}\int_\Omega u\,dx = 0 \; ?$$

a) Divergence form: $\nabla\cdot(\beta u)$

Assume: $\varepsilon\,\partial_n u = 0$ on $\Gamma\times I$ (insulation)

$\beta\cdot n = 0$ on $\Gamma\times I$ (no convection though boundary)

$f = 0$ (no heat source)

$\alpha = 0$ (no absorption)

Then we get:

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \dot{u}\,dx = \int_\Omega \big(\nabla\cdot(\varepsilon\nabla u) - \nabla\cdot(\beta u)\big)dx$$

Gauss theorem (B.13) $\int_\Omega \nabla\cdot v\,dx = \int_\Gamma v\cdot n\,ds$

$$= \int_\Gamma \big(\underbrace{\cancel{\varepsilon\nabla u\cdot n}}_{\varepsilon\partial_n u\,=\,0} - \underbrace{\cancel{(\beta u)\cdot n}}_{u(\beta\cdot n)\,=\,0}\big)ds = 0$$

$$\Rightarrow \int_\Omega u\,dx = 0 \text{ — conserved!}$$

b) Non-divergence form: $\beta \cdot \nabla u$

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \dot{u}\,dx = \int_\Omega \left(\nabla\cdot(\varepsilon\nabla u) - \beta\cdot\nabla u\right)dx$$

$$= \int_\Omega \Big(\underbrace{\nabla\cdot(\varepsilon\nabla u) - \nabla\cdot(\beta u)}_{=0 \text{ as above}} + (\nabla\cdot\beta)u\Big)dx$$

$$= \int_\Omega (\nabla\cdot\beta)u\,dx.$$

$\Rightarrow$ Temperature is conservative if $\nabla\cdot\beta = 0$ (or β divergence free)

Example: Problem 18.6. $\begin{cases} -\varepsilon u'' + u' = 0 & \text{in } \Omega = (0,1) \\ u(0)=1,\ u(1)=0 & (\Gamma_- = 0,\ \Gamma_+ = 1) \end{cases}$

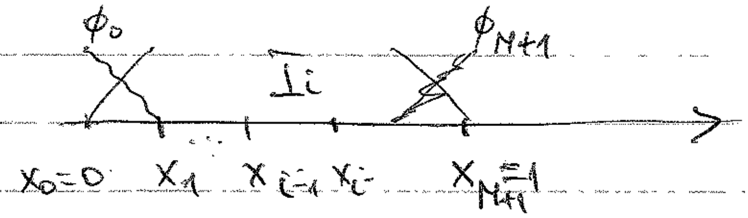
cG(1): Find $U_{h} \in V_{h} = \{ V ; cG(1) \}$

(GA) $\displaystyle\int_0^1 \varepsilon U' v' \, dx + \int_0^1 U' v \, dx = 0 \quad \forall v \in \hat{V}_h$

$V_h = \{ v \in H^1(0,1) : v(0) = 1 ; v(1) = 0 \}$

$\hat{V}_h = \{ v \in H^1(0,1) : v(0) = 0,\ v(0) = 0 \} = H_0^1(0,1)$

$\mathcal{T}_h$: uniform mesh with M interior points.



$h = \dfrac{1}{M+1}$

$U \in V_h \Rightarrow U = \displaystyle\sum_{j=1}^{M} \xi_j \phi_j(x) + \xi_0 \phi_0, \quad \xi_0 = 1$

Inset into (GA) we get the following:

$A \xi = b$, where

$$\begin{cases} A = (a_{ij}) = \displaystyle\int_0^1 \varepsilon \phi_j' \phi_i' \, dx + \int_0^1 \phi_j' \phi_i \, dx \\ b = (b_i) = -\displaystyle\int_0^1 \varepsilon u(0) \phi_0'(x) \phi_i'(x) dx + \int_0^1 u(0) \phi_0'(x) \phi_0(x) \end{cases}$$

$$\sum_{j=1}^{M} \xi_j \int_0^1 \varepsilon \varphi_j' \varphi_i' dx + \sum_{j=1}^{M} \xi_j \int_0^1 \varphi_j' \varphi_i dx =$$

$$= -\int_0^1 \varphi_0' \varphi_i' dx - \int_0^1 \varphi_0' \varphi_i dx$$

$$\Rightarrow \sum_{j=1}^{M} \xi_j \underbrace{\left(\int_0^1 \varepsilon \varphi_j' \varphi_i' dx + \int_0^1 \varphi_j' \varphi_i dx \right)}_{a_{ij}} = \underbrace{-\int_0^1 \varphi_0' \varphi_i' dx - \int_0^1 \varphi_0' \varphi_i dx}_{b_i}$$

$$a_{ii} = \int_{x_{i-1}}^{x_i} \left(\varepsilon \frac{1}{h} \frac{1}{h} + \frac{1}{h} \frac{x - x_{i-1}}{h} \right) dx + \int_{x_i}^{x_{i+1}} \varepsilon \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) + \left(-\frac{1}{h}\right) \frac{x_i - x}{h} \, dx$$

$$= \frac{\varepsilon}{h} + \frac{1}{2} + \frac{\varepsilon}{h} - \frac{1}{2} = \frac{2\varepsilon}{h}$$

$$a_{i-1,i} = \int_{x_{i-1}}^{x_i} \left(\varepsilon \left(-\frac{1}{h}\right) \frac{1}{h} + \left(-\frac{1}{h}\right) \frac{x - x_{i-1}}{h} \right) dx = -\frac{\varepsilon}{h} - \frac{1}{2}$$

$$a_{i,i+1} = \int_{x_i}^{x_{i+1}} \left(\varepsilon \frac{1}{h}\left(-\frac{1}{h}\right) + \frac{1}{h} \frac{x_i - x}{h} \right) dx = -\frac{\varepsilon}{h} + \frac{1}{2}$$

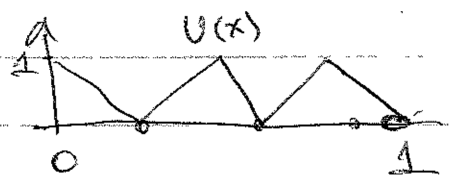
$$\Rightarrow a_{i-1,i} \neq a_{i,i+1} \Rightarrow \mathring{A} \text{ -nonsymmetric}$$

Descrete equation i:

$$\sum_{j=1}^{M} \xi_j a_{ij} = \xi_{i-1}\left(-\frac{\varepsilon}{h}-\frac{1}{2}\right)+\xi_i \frac{2\varepsilon}{h}+\xi_{i+1}\left(-\frac{\varepsilon}{h}+\frac{1}{2}\right)=0$$

if $\frac{\varepsilon}{h}$ large $\Rightarrow -\xi_{i-1}+2\xi_i-\xi_{i+1}=0$

if $\frac{\varepsilon}{h}$ small $\Rightarrow -\frac{1}{2}\xi_{i-1}+\frac{1}{2}\xi_{i+1}=0$

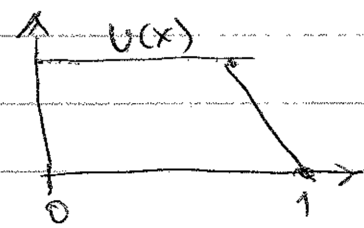


$\Rightarrow \xi_{i+1}=\xi_{i-1}$

solution oscilates!

<u>Stabilization</u> Choose $\varepsilon=\frac{h}{2}$

$\Rightarrow -\xi_{i-1}+\xi_i=0$ (upwing method)



<u>Conclusion</u>: ◻ GFEM works optimal for diffusion dominated problems;

◻ – " – non optimal for advection dominated problems;

◻ For nonsmooth exact solution U contains spurios oscillations if $\frac{\varepsilon}{h}$ small

◻ $\varepsilon \sim h$ works, but bad accuracy!

Streamline diffusion method

(DE) $Au = f$

(GA) Find $U \in V_h$: $(AU, v) = (f, v) \quad \forall v \in V_h$

(LS) Find $U \in V_h$: $\|AU - f\|^2 = \min_{v \in V_h} \|Av - f\|^2$

Corresponds to $(AU, Av) = (f, Av) \quad \forall v \in V_h$

(GLS) Weighted combination of (GA) & (LS)

Find $U \in V_h$: $(AU, v) + (\delta AU, Av) = (f, v) + (\delta f, Av) \quad \forall v \in V_h$

$(AU, v + \delta Av) = (f, v + \delta Av) \quad \forall v \in V_h$

If $U \in \hat{V}_h$, $v \in V_h$, $\hat{V}_h \neq V_h \Rightarrow$ GLS = PG

(SD) PG + artificial viscosity :

Find $U \in \hat{V}_h$:

$(AU, v + \delta Av) + (\hat{\varepsilon} \nabla U, \nabla v) = (f, v + \delta Av), \quad \forall v \in V_h$

$$\hat{\varepsilon} = \gamma_1 h^2 |R(U)|, \quad \forall v \in V_h$$

Assume $(Av, v) \geq c\|v\|^2$, $c > 0$

let $v = U$ in (SD) $\Rightarrow$

$(AU, U) + (\delta AU, AU) + (\hat{\varepsilon} \nabla U, \nabla U) = (f, U) + (\delta f, AU)$

$c\|U\|^2 + \|\sqrt{\delta} AU\|^2 + \|\sqrt{\hat{\varepsilon}} \nabla U\|^2 \leq \|f\| \|U\| + \|\sqrt{\delta} f\| \|\sqrt{\delta} AU\|$