Numerical stabilization of the finite element methods

Assume the following differential equation:

(DE) $\quad Au = f$, + boundary conditions and initial data.

Galerkin FEM states: Find $U \in V_h$ such that

(GFEM) $\quad (AU, v) = (f, v) \quad \forall v \in V_h$

where V_h is an apropriate finite element space.

The streamline diffusion method states: Find $U \in V_h$ such that

(SD) $\quad (AU, v + \delta Av) + (\varepsilon \nabla U, \nabla v) = (f, v + \delta Av) \quad \forall v \in V_h.$

Standard energy estimates

(GFEM): Take $v = U$ and assume that

$$(AU, U) \geq c\|U\|^2 \quad , \quad c > 0$$

then we get

$$(AU, U) = (f, U)$$

$$\Rightarrow \quad c\|U\|^2 \leq (f, U).$$

Using the fact that

$$(f, U) \leq \|f\|\|U\| \leq \frac{1}{2c}\|f\|^2 + \frac{c}{2}\|U\|^2$$

$$\Rightarrow \quad c\|U\|^2 \leq \frac{1}{2c}\|f\|^2 + \frac{c}{2}\|U\|^2$$

$$\Rightarrow \quad \frac{c}{2}\|U\|^2 \leq \frac{1}{2c}\|f\|^2 \Rightarrow \|U\|^2 \leq \frac{1}{c^2}\|f\|^2$$

That gives that the Galerkin approximation guarantees bound of U not ∇U and A.

(SD) Use the same assumption as above and $v = U$:

$$(AU, U) + (AU, \delta AU) + (\hat{\varepsilon}\nabla U, \nabla U) = (f, U) + (f, \delta AU)$$

$$\Rightarrow c\|U\|^2 + (\sqrt{\delta}AU, \sqrt{\delta}AU) + (\sqrt{\hat{\varepsilon}}\nabla U, \sqrt{\hat{\varepsilon}}\nabla U) \leq (f, U) + (\sqrt{\delta}f, \sqrt{\delta}AU)$$

$$\Rightarrow c\|U\|^2 + \|\sqrt{\delta}AU\|^2 + \|\sqrt{\hat{\varepsilon}}\nabla U\|^2 \leq \|f\|\|U\| + \|\sqrt{\delta}f\|\|\sqrt{\delta}AU\|$$

$$\leq \frac{c}{2}\|U\|^2 + \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta}f\|^2 + \frac{1}{2}\|\sqrt{\delta}AU\|^2$$

$$\Rightarrow \quad \frac{c}{2}\|U\|^2 + \frac{1}{2}\|\sqrt{\delta}\,AU\|^2 + \|\sqrt{\hat{\varepsilon}}\,\nabla U\|^2 \le \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta}\,f\|^2$$

This shows that $\|U\|^2$, $\|AU\|^2$ and $\|\nabla U\|^2$ are bounded when the (DE) is solved by the streamline diffusion method!