Courses/FEM/modules/stability

From Icarus

< Courses | FEM

Contents

Stability

Precondition

- Science
- Function approximation
- Galerkin's method

Theory

Galerkin's method for initial value ODE (timestepping)

We can write an ODE in general form

$$\begin{aligned} \dot{u} &= f(t,u) \\ u(0) &= u_0 \end{aligned}$$

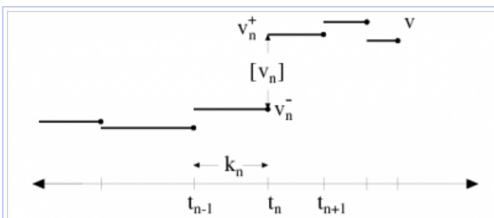
We choose a model problem

$$\begin{aligned} \dot{u} &= -a(t)u + f(t) \\ u(0) &= u_0 \end{aligned}$$

dG(0) / Backward Euler

We introduce the discontinuous Galerkin method with U represented by piecewise constant polynomials dG(0).

We seek $U \in W_k$ with $U_0^- = u_0$, W_k space of piecewise constants



Example function and notation for dG(0)

$$\int_{t_{n-1}}^{t_n} \dot{U}v + aUv - fvdt + (U_n - U_{n-1})v = 0, \quad \forall v \in W_k \Rightarrow$$

$$U_n = U_{n-1} + \int_{t_{n-1}}^{t_n} -a(t)U(t) + f(t)dt$$

$$U = \sum_{i=1}^{N} \xi_i \phi_i, \quad \phi_i = 1, \quad k_n = t_n - t_{n-1} \Rightarrow$$

$$\xi_n = \xi_{n-1} + \int_{t_{n-1}}^{t_n} -a(t_n)\xi_n\phi_n + f(t_n)dt$$

Note that $\dot{U} = 0$ on a time interval $[t_{n-1}, t_n]$.

By computing the integral with the right-point quadrature rule, we get the familiar backward Euler time-stepping method (with an additional quadrature error, which we can omit in implementation, but should take into account in detailed error analysis)

$$\xi_n = \xi_{n-1} + k_n(-a(t_n)\xi_n + f(t_n)) + E_q$$

cG(1) / Crank-Nicolson

We introduce the continuous Galerkin method with U represented by piecewise linear polynomials cG(1).

We seek $U \in V_k$ with $U(0) = u_0$, V_k space of piecewise linears

$$\int_{t_{n-1}}^{t_n} \dot{U}v + aUv - fvdt = 0, \quad \forall v \in W_k \Rightarrow$$
$$U_n = U_{n-1} + \int_{t_{n-1}}^{t_n} -a(t)U(t) + f(t)dt$$
$$U = \sum_{i=1}^{N} \xi_i \phi_i, \quad \phi_i = \frac{t - t_{i-1}}{t_i - t_{i-1}}, \quad k_n = t_n - t_{n-1} \Rightarrow$$
$$\xi_n = \xi_{n-1} + \int_{t_{n-1}}^{t_n} -a(t_n)(\xi_{n-1}\phi_{n-1} + \xi_n\phi_n) + f(t_n)dt$$

By computing the integral with the trapezoid quadrature rule, we get the familiar Crank-Nicolson time-stepping method (with an additional quadrature error, which we can omit in implementation, but should take into account in detailed error analysis)

$$\xi_n = \xi_{n-1} + \frac{1}{2}k_n(-a(t_{n-1})\xi_{n-1} - a(t_n)\xi_n + f(t_{n-1}) + f(t_n)) + E_q$$

Stability

We define the *stability* of the solution u (or derivatives) of an equation as sensitivity to perturbations in data f (source) and $u(0)$ (initial value).

We are seeking bounds (estimates) of type

$$\begin{aligned} \|u\| &\leq S\|f\| \\ \|u\| &\leq S\|u(0)\| \\ \|U\| &\leq S\|f\| \\ \|U\| &\leq S\|u(0)\| \end{aligned}$$

for the exact and discrete solution respectively, where S is a constant/factor which doesn't depend on u or U.

Stability for model problems

We look at a collection of model problems:

Heat equation

$$\begin{aligned} \dot{u} - \Delta u &= f(t,x) \\ u(0,x) &= u_0(x) \\ u(t,x) &= 0, \quad x \in \Gamma \end{aligned}$$

Wave equation

$$\begin{aligned} \ddot{u} - \Delta u &= f(t,x) \\ u(0,x) &= u_0(x) \\ \dot{u}(0,x) &= \dot{u}_0(x) \\ u(t,x) &= 0, \quad x \in \Gamma \end{aligned}$$

or equivalently (to help analysis), with $u_1 = u$

$$\begin{aligned} \dot{u_1} - \Delta u_2 &= 0 \\ \Delta \dot{u_2} - \Delta u_1 &= f(t,x) \\ u(0,x) &= u_0(x) \\ \dot{u}(0,x) &= \dot{u}_0(x) \\ u(t,x) &= 0, \quad x \in \Gamma \end{aligned}$$

Stability estimates for the heat equation

Assume $f = 0$.

Use identity

$$\frac{1}{2}D_t\|u\|^2 = \frac{1}{2}D_t\int_\Omega uudx = \frac{1}{2}\int_\Omega(\dot{u}u + u\dot{u})dx = \int_\Omega \dot{u}udx = (\dot{u}, u)$$

Weak form of heat equation:

Multiply with v and integrate in space and time

$$\int_0^T\int_\Omega(\dot{u} - \Delta u)vdxdt = \int_0^T(\dot{u}, v) + (\nabla u, \nabla v)dt = 0$$

Galerkin's method cG(1)dG(0) (backward Euler) for the heat equation

$$(U_n, v) = (U_{n-1}, v) - k_n(\nabla U(t_n, x), \nabla v)), \quad \forall v \in \bar{W}_k$$

After solving for U_n, this statement is true for all v, i.e. we have a theorem for every choice of v.

1. Choose $v = u$ in weak form
2. $$\int_0^T(\dot{u}, u) + (\nabla u, \nabla u)dt = 0 \Rightarrow$$
3. $$\int_0^T \frac{1}{2}\|u\|^2 + \|\nabla u\|^2 dt = 0 \Rightarrow$$
4. $$\|u(T)\|^2 + 2\int_0^T\|\nabla u\|^2 dt = \|u(0)\|^2$$

   Dissipation of temperature/energy (u).
5. Choose $v = U_n$ in discrete equation
6. $$(U_n, U_n) = (U_{n-1}, U_n) - k_n(\nabla U_n, \nabla U_n)) \Rightarrow$$
7. $$(U_n - U_{n-1}, U_n) + k_n(\nabla U_n, \nabla U_n)) = 0 \Rightarrow$$

   Use $U_n = \frac{1}{2}(U_n - U_{n-1}) + \frac{1}{2}(U_n + U_{n-1})$
8. $$(U_n - U_{n-1}, \frac{1}{2}(U_n - U_{n-1})) + (U_n - U_{n-1}, \frac{1}{2}(U_n + U_{n-1})) + k_n(\nabla U_n, \nabla U_n)) = 0 \Rightarrow$$
9. $$\frac{1}{2}\|U_n - U_{n-1}\|^2 + \frac{1}{2}\|U_n\|^2 + k_n\|\nabla U_n\|^2 = \frac{1}{2}\|U_{n-1}\|^2 \Rightarrow$$
10. $$\|U_n\|^2 + 2k\|\nabla U\|^2 \leq \|U_{n-1}\|^2$$

    Similar statement for discrete solution.
11. Directly from 2
12. $$\|U_n\|^2 \leq \|U_{n-1}\|^2$$

    Discrete solution does not grow for any time step.

Stability estimates for the wave equation

Assume $f = 0$.

Weak form of wave equation:

Multiply with v_1, v_2 and integrate in space and time

$$\int_0^T\int_\Omega(\dot{u_1}v_1 - \Delta u_2 v_1 - \Delta\dot{u_2}v_2 + \Delta u_1 v_2)dxdt = 0 \Rightarrow$$

$\int_0^T((\dot{u_1}, u_1) - (\Delta u_2, u_1) + (-\Delta\dot{u_2}, u_2) + (\Delta u_1, u_2))dt = 0$

Galerkin's method cG(1)cG(1) (Crank-Nicolson) for the wave equation

$$(U1_n, v_1) - (U1_{n-1}, v_1) + \frac{1}{2}k_n(\nabla U2_n, \nabla v_1)) + \frac{1}{2}k_n(\nabla U2_{n-1}, \nabla v_1)) - (\nabla U2_n, \nabla v_2) + (\nabla U2_{n-1}, \nabla v_2) + \frac{1}{2}k_n(\nabla U1_n, \nabla v_2) + \frac{1}{2}k_n(\nabla U1_{n-}$$

1. Multiply with v_1, v_2 and integrate in space

2. $$(\dot{u_1}, u_1) - (\Delta u_2, u_1) + (-\Delta \dot{u_2}, u_2) + (\Delta u_1, u_2) = 0 \Rightarrow$$
3. $$(\dot{u_1}, u_1) + (\nabla u_2, \nabla u_1) + (\nabla \dot{u_2}, \nabla u_2) - (\nabla u_1, \nabla u_2) = 0 \Rightarrow$$
4. $$(\dot{u_1}, u_1) + (\nabla \dot{u_2}, \nabla u_2) = 0 \Rightarrow$$
5. $$D_t(\|\dot{u}\|^2 + \|\nabla u\|^2) = 0$$

   Total energy conserved.
6. Choose $v_1 = U1_n, v_2 = U2_n$

   Same process as for heat equation
7. $$\|U_n\|\|^2 + \|\nabla U_n\|^2 = \|U_{n-1}\|^2 + \|\nabla U_{n-1}\|^2$$

   Also total energy for discrete solution conserved.

Stabilization / Streamline-diffusion

We examine the convection-diffusion equation

$$\begin{aligned} \dot{u} + \nabla \cdot (\beta u) - \nabla \cdot (\epsilon \nabla u) + \alpha u &= f(t,x) \\ u(0,x) &= u_0(x) \\ (\nabla u(t,x) \cdot n) &= 0, \quad x \in \Gamma_N \\ u(t,x) &= 0, \quad x \in \Gamma_D \end{aligned}$$

We can write

$$\nabla \cdot (\beta u) = \beta \cdot \nabla u + (\nabla \cdot \beta) u$$

If $\nabla \cdot \beta = 0$ (divergence-free) we have

$$\nabla \cdot (\beta u) = \beta \cdot \nabla u$$

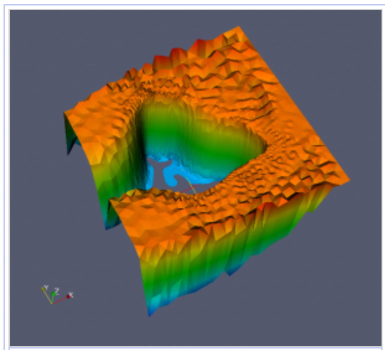
For this discussion we can also assume for simplicity

$$\dot{u} = 0$$

Standard Galerkin

Galerkin's method for convection-diffusion

$$(\beta \cdot \nabla U, v) + (\epsilon \nabla U, \nabla v) + (\alpha U, v) - (f, v) = 0, \quad \forall v \in V_h$$



Example of unstable solution for standard Galerkin

Examine stability by choosing $v = U$:

Use Young's inequality

$$0 \leq (a - cb)^2 = a^2 - 2cab + c^2 b^2 \Rightarrow$$

$ab \leq \frac{a^2}{2c} + \frac{c}{2} b^2$

We can define the operator $Aw = \beta \cdot w - \Delta w + \alpha w$

We assume $(Av, v) \geq c\|v\|^2$.

$(\beta \cdot \nabla U, U) + (\epsilon \nabla U, \nabla U) + (\alpha U, U) - (f, U) = 0 \Rightarrow$

$c\|U\|^2 \leq \|f\|\|U\| \Rightarrow$

$c\|U\|^2 \leq \frac{c}{2}\|U\|^2 + \frac{1}{2c}\|f\|^2 \Rightarrow$

$c\|U\|^2 \leq \frac{1}{c}\|f\|^2$

If diffusion coefficient ϵ is small (like 0), ∇U can grow large, while U cannot. I.e. we have no control of derivatives of U, only of U itself.

Streamline diffusion / Least squares

Streamline diffusion / Galerkin least squares method for convection-diffusion

$$(AU, v + \delta A v) - (f, v + \delta A v) = 0, \quad \forall v \in V_h, \quad \delta = \frac{h}{|\beta|}$$

Examine stability by choosing $v = U$

$$(AU, U + \delta AU) - (f, U + \delta AU) = 0 \Rightarrow$$

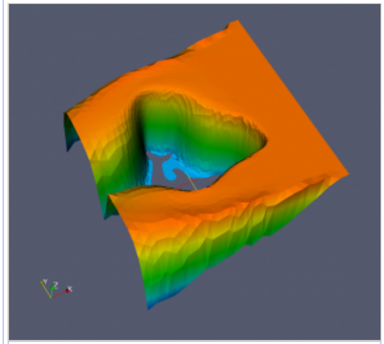
$$c\|U\|^2 + \|\sqrt{\delta}AU\|^2 \leq \|f\|\|U\| + \|\sqrt{\delta}f\|\|\sqrt{\delta}AU\| \Rightarrow$$

$$c\|U\|^2 + \|\sqrt{\delta}\beta \cdot \nabla U\|^2 \leq \frac{c}{2}\|U\|^2 + \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta}f\|^2 + \frac{1}{2}\|\sqrt{\delta}\beta \cdot \nabla U\| \Rightarrow$$

$$c\|U\|^2 + \|\sqrt{\delta}AU\|^2 \leq \frac{c}{2}\|U\|^2 + \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta}f\|^2 + \frac{1}{2}\|\sqrt{\delta}AU\| \Rightarrow$$

$$\frac{c}{2}\|U\|^2 + \frac{1}{2}\|\sqrt{\delta}AU\|^2 \leq \frac{1}{2c}\|f\|^2 + \frac{1}{2}\|\sqrt{\delta}f\|^2 \Rightarrow$$

$$\|U\|^2 + \|\sqrt{\delta}AU\|^2 \leq C\|f\|^2$$



Example of stabilized solution with streamline-diffusion

Even if diffusion constant ϵ is small (like 0), $\|\sqrt{\delta}\beta \cdot \nabla U\|^2$ cannot grow large. I.e. we have control of derivatives of U (in the streamline direction).

We typically use the simple variant (where we only stabilize the convection term)

$$(\beta \cdot \nabla U, v + \delta(\beta \cdot \nabla v)) + (\epsilon \nabla U, \nabla v) + (\alpha U, v) - (f, v) = 0, \quad \forall v \in V_h$$

The streamline-diffusion method also includes a shock-capturing term $(\hat{\epsilon}\nabla U, \nabla v)$ for capturing discontinuities in the solution. We omit this discussion here, and refer to CDE chapter 18.

Software

Postcondition

You should now be familiar with:

- What a stability estimate is
- Stability estimates for the heat equation
- The streamline-diffusion method

Exercises

9.4, 9.5, 9.14, 9.43, 10.18, 10.21, 10.28, 16.18, 17.19, 17.20, 17.27, 18.7, 18.9

Examination

1.1

You are given an implementation of the standard Galerkin method applied to the convection-diffusion equation. Modify the implementation to implement the streamline-diffusion method and discuss the solutions for the two methods, why does one work better than the other? What is the relation to stability?

The necessary files are:

- http://www.icarusmath.com/icarus/images/Streamline.py
- http://www.icarusmath.com/icarus/images/Dolfin-2.xml.gz
- http://www.icarusmath.com/icarus/images/Subdomains.xml.gz
- http://www.icarusmath.com/icarus/images/Velocity.xml.gz

Note: For the implementation in this question we define $\alpha = 0$ and we can use the simple variant where we only look at the convection term in the stabilization. For simplicity you can approximate $|\beta| = 1$.