# Written Examination: Mathematical Models, Analysis and Simulation, Part I

Stockholm, December 14, 2005

**Closed book examination** 5 hours. A sum of credits of 50, homework included, is certainly enough to pass. The results will be announced no later than December 23, 2005.

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| Read all the questions first. They are not necessarily sorted in order of ascending difficulty. Work the easy ones first. Good luck! |

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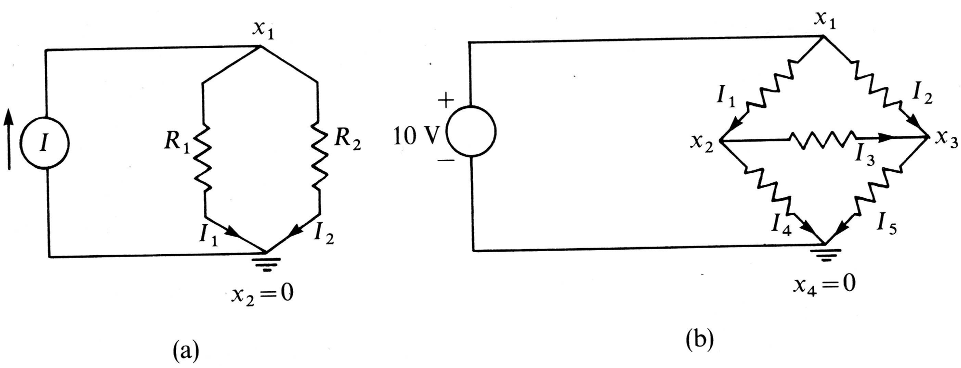
1. Consider the two networks below:




   (a) By using the network approach, show that the network resistance of two resistors in parallel is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}!$$

   In order to show that, use circuit (a). Write down the linear system for determining $I_1, I_2, x_1$ by using Ohm's law and Kirchhoff's current law! Solve it. Then $R = x_1/I$. **(3)**

(b) For the circuit in Figure (b) the voltage source fixes $x_1 = 10$. Write down the five equations $e = -A_0 x$, and set $x_1 = 10, x_4 = 0$ to reach $e = b - Ax$; $A$ has two columns multiplying $x_2$ and $x_3$. Similarly $A^T y = f$ has only the two equations $I_1 - I_3 - I_4 = 0$ and $I_2 + I_3 - I_5 = 0$, at nodes 2 and 3. With unit resistors $R = 1$ solve the equilibrium equations for $y$ and $x$. What is the current $I_3$ and what is the network resistance $V/I$ to the voltage $V = 10$?**(3)**

2. The phenomenon we consider here is the appearance of solitary water waves in a narrow channel. Once a single wave has formed, it will survive a long period of time almost unperturbed. *Korteweg and deVries* derived a model for waves in a long straight channel such that only one space dimension remains. Their equation is

$$y_t = -\sqrt{\frac{g}{h}} \frac{\partial}{\partial x}\left(hy + \frac{3}{4}y^2 + \frac{1}{2}\sigma y_{xx}\right),$$

where $g$ is the gravitational constant, $h$ the water depth if the water is at rest, and $\sigma$ is given by

$$\sigma = \frac{1}{3}d^3 - \frac{\mu d}{\rho g}.$$

Here, $\mu$ is the surface tension, $\rho$ the water density, and $d$ the depth of the channel. The equation is defined for $-\infty < x < \infty$ and $t \geq 0$. The solution $y$ is the elevation of the wave above rest. According to their assumptions, the value of $y$ and all of its derivatives vanishes at $\pm\infty$. The solitary wave solutions are described by *traveling wave solutions*, $y(x,t) = w(x - Vt)$ with the unknown velocity $V > 0$.

(a) Show that by a proper choice of the length and time scales $L$ and $T$, $y = L\eta$, $x = L\xi$, and $t = T\tau$, the equation becomes

$$\eta_t = -\frac{\partial}{\partial \xi}\left(\eta + \frac{3}{4}\eta^2 + \frac{1}{2}a\eta_{\xi\xi}\right).$$

What are $T$, $L$, and $a$? **(2)**

(b) We are now interested in traveling wave solutions $\eta(\xi,\tau) = f(\xi - v\tau)$. Derive the 3rd order ordinary differential equation for $f(s)$ with $s = \xi - v\tau$. **(2)**

Using the boundary conditions, the latter equation can be integrated once such that only a 2nd order equation remains. Derive it! **(1)**

(c) If you have not done (b) you may obtain the proper equation from the supervisor such that you can continue with the next part.

For the following assume that all constants are positive. Show by rescaling that this equation is equivalent to

$$y'' = cy - \frac{3}{2}y^2$$

**(2)**

Compute the stationary points and determine their stability. Sketch the local phase planes around the critical points, including distinguished directions, when relevant. **(4)**

3. Consider the boundary value problem

$$\begin{aligned} -u'' &= 1, \quad 0 < x < 1, \\ u(0) &= 1, \quad u'(1) = 1. \end{aligned}$$

   (a) Find the weak formulation. Specify the bilinear form $a(u,v)$, the right-hand side $L(v)$, and the function space $V$ precisely. **(3)**

   (b) Approximate the solution by a cubic polynomial using Galerkin's method! **(3)**

   (c) Use two linear finite elements ("roof" functions) with $x_0 = 0, x_1 = 1/3, x_2 = 1$ to approximate the solution by the finite element method. What is the dimension of the mass matrix? Assemble the load vector. **(3)**

   (d) What is $u_h'(1)$? **(2)**

4. For the solution of the normal equations $A^T Ax = A^T b$, the following iterative method can be used:

$$x_{k+1} = x_k - hA^T(Ax_k - b), \quad h > 0.$$

   The matrix $A$ has dimension $m \times n$ with $m > n$.

   (a) Let $P = A^T A$ have the eigenvalues $\lambda_i$, $i = 1, 2, \ldots, n$. Show that they are nonnegative. **(2)**

   (b) Let the solution of the normal equations be $x^*$. Let $e_k = x^* - x_k$. Show that $e_{k+1} = B(h)e_k$, where $B(h) = I - hP$. **(2)**

   (c) Assume that a lower bound $a > 0$ and an upper bound $b$ are known for $\lambda_i$. Give upper and lower bounds for the eigenvalues of $B(h)$. **(1)**

   (d) Prove that the iteration converges for $h = 1/b$. **(2)**

   (e) With the information given in (c) give the best value of $h$, i.e. the value for which the convergence is fastest. **(3)**

5. Heat transfer by longitudinal convection ($V > 0$) and transversal conduction is described by

$$\begin{gathered} T_t + VT_x = -aT, \quad 0 \le x < \infty, \quad t \ge 0, \quad a > 0 \\ T(0,t) = f(t), \quad T(x,0) = g(x). \end{gathered}$$

(a) Write the characteristic equations ($dX/dt = \ldots, dT(X(t),t)/dt = \ldots$). For what part of the positive quadrant $x > 0, t > 0$ is the solution completely determined by the *initial condition*? The *boundary condition* at $x = 0$?
Write the exact solution in terms of $f$ and $g$. **(4)**

(b) The forward in time, central in space scheme is applied with space-step $h$ and time-step $\tau$, $T_i^n \approx T(ih, n\tau)$ to give

$$(T_i^{n+1} - T_i^n)/\tau + V/(2h)(T_{i+1}^n - T_{i-1}^n) = -aT_i^n.$$

Compute the von Neumann stability analysis growth factor $G = G(r, \theta, \tau)$. Sketch the locus of $|G| = 1$ as $\theta$ varies from $-\pi$ to $\pi$ for a few (strategically chosen) values of $a\tau$ and the Courant number $r = V\tau/h$. Also sketch the unit circle! **(3)**

Show that $a\tau < 2$ is necessary for stability. Derive additionally a complete expression for the stability limit on $\tau$. **(2)**

6. The inverse of $B = I - vw^T$ has the form $B^{-1} = I - cvw^T$.

(a) Show that this is true and find the number $c$. Under what conditions on $v$ and $w$ is $B$ not invertible? **(2)**

(b) Consider the more general matrix $B = A - vw^T$ with invertible $A$, find the inverse matrix. When is $B$ not invertible? **(2)**

(c) If you subtract 1 from the first entry $a_{11}$ of $A$, what matrix is subtracted from $A^{-1}$? In $A^{-1}$ let $q$ be the first column, $r^T$ be the first row, and $s$ be the first entry. **(4)**