Written Examination: Mathematical Models, Analysis and Simulation, Part I

Stockholm, December 14, 2005

Solutions

1.

(a) The network equations give a system

$$\begin{pmatrix} C^{-1} & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}$$

where A denotes the (reduced) incidence matrix, y the branch currents, x the node potentials, b the voltage sources, and f the current sources. In the present example:

$$A = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, C = \begin{pmatrix} 1/R_1 & \\ & 1/R_2 \end{pmatrix}, x = x_1, y = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, f = -I.$$

Solving the system yields

$$x_1 = \frac{I}{\frac{1}{R_1} + \frac{1}{R_2}}$$

(b) Similarly as before, we have

$$A_0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, b = \begin{pmatrix} 10 \\ 10 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and C the unit matrix. Solving the system yields $I_3 = 0$ and the total resistance, $R_{tot} = R$.

2.

(a) $L = h$, $T = \sqrt{h/g}$, $a = \sigma/h^3$.

(b) $vf' = f' + \frac{3}{2}ff' + \frac{1}{2}af'''$
$a\frac{d^2y}{ds^2} = by - \frac{3}{2}y^2$

(c) The new scaling is $s_* = s/\sqrt{\sigma}$. This leads to the system

$$\begin{pmatrix} y' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ cy - \frac{3}{2}y^2 \end{pmatrix}.$$

The system has two stationary points, $(0,0)$ and $(0, \frac{2}{3}c)$.Using the Jacobian and computing the eigenvalues we have

- $(0,0)$: $\lambda = \pm\sqrt{c}$, which gives a saddle point
- $(0, \frac{2}{3}c)$: $\lambda = \pm i\sqrt{c}$, which is a center

3. Let $H^1(0,1)$ denote the Sobolev space of all square integrable functions with square integrable first derivatives.

(a) Let $V = \{v \in H^1(0,1) | v(0) = 0\}$ and $W = \{u | u - 1 \in V\}$. The weak formulation is: Find $u \in W$ such that $a(u,v) = L(v)$ for all $v \in V$ where

$$a(u,v) = \int_0^1 u'v'dx, L(v) = \int_0^1 vdx + v(1).$$

(b) ansatz: $u(x) = 1 + ax + bx^2 + cx^3$. Applying Galerkin method gives $u(x) = -\frac{x^2}{2} + 2x + 1$.

(c) The dimension depends on the handling of the boundary conditions. It may be 2 or 3. The load vector is $L = (1/2, 4/3)^T$.

(d) $u'_h = 1 + h/2$, where $h = 1 - 1/3 = 2/3$.

4.

(a) Let λ be an eigenvalue of P. Then it holds $A^TAx = \lambda x$. Hence $x^TA^TAx = \lambda x^Tx$ and $\|Ax\|^2 = \lambda\|x\|^2$. This yields $\lambda \geq 0$.

(b) It holds $x^* = x^* - hA^T(Ax^* - b)$. Subtracting this from the recursion: $e_{k+1} = e_k - hA^T(Ae_k - 0) = B(h)e_k$.

(c) The eigenvalues μ_i of $B(h)$ are $1 - h\lambda_1$. Hence, $1 - ha \geq \mu_i \geq 1 - hb$.

(d) Because of (c), $1 - \frac{a}{b} \geq \mu_i \geq 0$. Since $0 < a \leq b$, $\|B(h)\| < 1$.

(e) $h = 2/(a+b)$.

5. This problem is already given on 18/1-2003

6.

(a) $I = (I - vw^T)(I - cvw^T) = I - vw^T - cvw^T + cvw^T vw^T = I - (c + 1 - \alpha c)vw^T$ where $\alpha = w^T v$. This leads to $c = 1/(\alpha - 1)$. The matrix is singular iff $\alpha = 1$.

(b) $B = A - vw^T = A(I - A^{-1}vw^T)$. Using (a), $B^{-1} = (I - cA^{-1}vw^T)A^{-1}$ where $c = 1/(\alpha - 1)$ and $\alpha = w^T A^{-1} v$. This matrix is singular iff $w^T A^{-1} v = 1$.

(c) Subtraction of the first entry is obtained by choosing $v = w = e_1$the first unit vector. The results of (b) give $\alpha = s$ such that the modification matrix becomes $\frac{1}{s-1}qr^T$.