Written Examination: Mathematical Models, Analysis and Simulation, Part I

Stockholm, January 17, 2006

Closed book examination 5 hours. A sum of credits of 50, homework included, is certainly enough to pass. The results will be announced no later than February 7, 2006.

Read all the questions first. They are not necessarily sorted in order of ascending difficulty. Work the easy ones first. Good luck!

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1. Given the following $n \times n$-matrix A with the elements

$$a_{ii} = a, \quad i = 1, \dots, n, \quad a_{ij} = 1, \quad i \neq j.$$

   (a) What is the rank of A? Note that the results depends on a. Motivate your answer. **(2)**

   (b) Which are the eigenvalues of A? **(3)**

   (c) For which values of a is A positive definite? **(2)**

2. A is a real symmetric $n \times n$-matrix with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The following iterative scheme is used for solving $Ax = b$,

$$x_{k+1} = x_k - \alpha r_k, \quad r_k = Ax_k - b.$$

   Let $e_k = A^{-1}b - x_k$ and $\kappa = \lambda_n/\lambda_1$, the condition number of A.

   (a) Show that $e_{k+1} = Be_k$ with $B = I - \alpha A$. What are the eigenvalues of B? **(3)**

   (b) Show that

$$\frac{e_{k+1}^T e_{k+1}}{e_k^T e_k} \leq \max_{1 \leq l \leq n} (1 - \alpha\lambda_l)^2.$$

   (3)

(c) Show that one can choose α such that

$$\|e_{k+1}\|_2 \le \frac{\kappa - 1}{\kappa + 1}\|e_k\|_2$$

(4)

3. According to Braun, reptiles, mammals, and plants on the island of Komodo have populations governed by

$$\begin{aligned} u_r' &= -au_r - bu_r u_m + cu_r u_p \\ u_m' &= -du_m + eu_r u_m \\ u_p' &= fu_p - gu_p{}^2 - hu_r u_p. \end{aligned}$$

All constants a, b, c, d, e, f, g, h are assumed to be positive.

(a) Who is eating whom? **(1)**

(b) Find the equilibrium solutions! Which of them are biologically meaningful? **(3)**

(c) Show that the positive octant $D = \{(u_r, u_m, u_p) | u_i \ge 0\}$ is positively invariant. **(2)**

(d) Is the origo stable? Sketch the phase plots near the origo for the cases that there are (i) no reptiles, (ii) no mammals, and (iii) no plants on the island. **(4)**

4. Consider the boundary value problem

$$\begin{aligned} -u'' + u = 1, \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{aligned}$$

(a) Find the weak formulation. Specify the bilinear form $a(u,v)$, the right-hand side $L(v)$, and the function space V precisely. **(2)**

(b) Approximate the solution by the Galerkin method using the ansatz $u_h(x) = c\sin(\pi x)$. **(2)**

(c) Use now piecewise linear finite elements with $x_0 = 0, x_1 = 1/2, x_2 = 1$. What are the approximate values for $u_h(1/2)$ in both cases (b) and (c)? How large is the error? The exact solution is $u(1/2) \approx 0.11318111602993$. **(3)**

5. Consider the conservation law $u_t + (Q(u))_x = 0$ where $Q(u) = u(1-u)$.

(a) Formulate the upstream method for the initial value problem $u(x,0) = f(x)$, $-\infty < x < \infty$ with $f(x) > 0$. **(2)**

(b) Write the equations for the characteristics and show that (i) u is constant along a characteristic, (ii) the characteristics are straight lines. **(3)**

(c) At $t = 0$, the concentration has a discontinuity at $x = 0$:

$$f(x) = 1/4, x \leq 0, \quad f(x) = 3/4, x > 0.$$

Find the shock speed and plot the solution at $t = 1$. **(2)**

(d) Suppose that $Q(u) = u(1 - u - u_x/u)$. You may assume that $u > 0$. Sketch how this would change the plot in (c). **(2)**

6. Given a least-squares problem with linear constraints

$$\min_{Cx=c} \frac{1}{2}\|Ax - a\|_2^2,$$

where A is an $m \times n$-matrix, $m > n$, having full rank n, and C is $p \times n$ with $p < n$ and rank p. The argument for which the minimum is taken is denoted by $\hat{x}$.

(a) Show that $\hat{x}$ satisfies the following linear system of equations

$$\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T a \\ c \end{pmatrix}$$

where λ is the Lagrange multiplier. **(3)**

(b) The matrix in (a) is obviously symmetric. Is it also positive definite? Give a proof or give a counterexample. **(3)**

(c) If the linear system in (a) is solved we can write the solution as

$$\hat{x} = Pa + Qc.$$

Give the explicit form of the two matrices P and Q as functions of A and C. **(4)**