Written Examination: Mathematical Models, Analysis and Simulation, Part I

Stockholm, January 17, 2006

Solutions

1. It holds $A = (a-1)I + ee^T$ where $e = (1,\dots,1)^T$.

   (a) It holds $Ax = 0$ iff $(a-1)x + \alpha e = 0$, $\alpha = e^T x$. This can only be true for $x \neq 0$ if $x = e$, $1 - a = \alpha = n$ unless $a = 1$. Hence,

   - $a = 1$: rank $A = 1$
   - $a \neq 1 - n$: rank $A = n$
   - $a = 1 - n$: the dimension of the nullspace is one such that the rank is $n - 1$.

   (b) let λ be an eigenvalue, x be an eigenvector. Then as above: $(a-1)x + ee^T x = \lambda x$ such that $(a - 1 - \lambda)x + \alpha e = 0$. We have agian two cases:

   - $\alpha = 0$: $\lambda = a - 1$.
   - $\alpha \neq 0$: $x = e$, $\alpha = n$, $a - 1 - \lambda + n = 0$, $\lambda = n + a - 1$.

   (c) A is positive definite iff all eigenvalues are larger than one. This is equivalent to $a > 1$.

2. Let $x_* = A^{-1}b$ the solution. Then $e_k = x_* - x_k$.

   (a) It holds $r_* = Ax_* - b = 0$ such that $x_* = x_* - \alpha r_*$. By subtraction of this equality from the recursion we obtain $e_{k+1} = e_k - \alpha(r_* - r_k) = e_k - \alpha A e_k = Be_k$. Moreover, the eigenvalues of B are $\mu_l = 1 - \alpha\lambda_l$.

   (b) According to the lecture: $\|e_{k+1}\| \leq \max_{1\leq l\leq n} |\mu_l| \|e_k\|$. Observing $\|x\|^2 = x^T x$, we obtain the desired result.

   (c) Choose $\alpha = 2/(\lambda_1 + \lambda_n)$. This leads to $\max_{1\leq l\leq 1} |\mu_l| = \max(|\mu_1|, |\mu_n|)$. With the given α, $\mu_1 = (\lambda_n - \lambda_1)/(\lambda_1 + \lambda_n)$ and $\mu_n = -\mu_1$. Inserting these expressions gives the desired result.

3.

(a) The first equation indicates that the number of reptiles decreases if the number of mammals increases. In a similar way, the third equation indicates a decrease in plants if the number of reptiles increases. Hence, mammals eat reptiles who in turn eat plants.

(b) Set $u_r' = u_m' = u_p' = 0$ and solve the resulting nonlinear system. There are exactly five solutions:

no	u_r	u_m	u_p
1	0	0	0
2	0	0	f/g
3	$(fc-ag)/(ch)$	0	a/c
4	d/e	$-a/b$	0
5	d/e	$(cu_p-a)/b$	$(ef-hd)/(eg)$

These solutions are biologically meaningful if they are nonnegative. So 1 and 2 are ok while 4 is not. If 3 and 5 are meaningful depends on the values of the constants.

(c) The eigenvalues of the Jacobian at the origo are $-a$, $-d$ and f. Hence, the origo is not stable.

4.

(a) Since we have homogeneous Dirichlet boundary conditons on both sides $V = H_0^1(0,1)$, the Sobolev space of once differentiable functions with square integrable derivative which fulfill the homogeneous boundary conditions. So the weak form is: Find a function $u \in V$ such that $a(u,v) = L(v)$ for all test functions $v \in V$. Here,

$$a(u,v) = \int_0^1 (u'v' + uv)dx, L(v) = \int_0^1 v dx$$

(b) Using the ansatz $u_h(x) = c\sin(\pi x)$ we can choose the test function $v_{h,1} = \sin(\pi x)$. This gives

$$a(u_h, v_{h,1}) \quad = \quad c\int_0^1 (\pi^2\cos^2(\pi x) + \sin^2(\pi x))dx = c(\frac{1}{2}(\pi^2 - 1) + 1),$$

and

$$L(v_{h,1}) = \int_0^1 \sin(\pi x)dx = \frac{2}{\pi}$$

Hence, $c = \frac{2}{\pi}/[\frac{1}{2}(\pi^2 - 1) + 1]$.

(c) The ansatz is now $u_h(x) = c\phi_1(x)$ where

$$\phi_1(x) = \begin{cases} 2x, & 0 < x < 1/2 \\ 2-2x, & 1/2 < x < 1 \end{cases}$$

This gives

$$a(c\phi_1, \phi_1) = c13/3, L(\phi_1) = 1/2$$

such that $c = 3/26$. We obtain $u_h(1/2) = 0.1171$ in case (b) and $u_h(1/2) = 0.11538$ in case (c).

5.

(a) The characteristic equations are

$$\frac{dx}{dt} = 1 - 2u, \frac{du(x(t),t)}{dt} = 0.$$

We see immediately from the second equation that u is constant along each characteristic. Using this information in the first equation shows that $dx/dt = const$ such that the characteristics are straight lines.

(b) The Rankine-Hugoniot condition for the shock speed is

$$\frac{ds}{dt} = \frac{Q(u^+) - Q(u^-)}{u^+ - u^-} = 0$$

such that the shock location does not change. Hence, the solution at $t = 1$ is the same as that at $t = 0$ that is f.

(c) Far away from the shock, u_x is neligible such that the solution will not be perturbed. However, near the discontinuity, the shock will be smeared out.

6.

(a) Formulate the Lagrange function $L(x, \lambda) = \frac{1}{2}\|Ax - a\|^2 + \lambda^T(Cx - c)$. Differentiation of this expression with respect to x gives $A^T A\hat{x} + C^T\lambda = A^T a$ while differentiation with respect to λ yields $Cx = c$.

(b) The matrix is indefinite. Take for example

$$\begin{pmatrix} 1 & a \\ a & 0 \end{pmatrix}, a = \sqrt{3}/2.$$

Then $\lambda_1 = -3/2$, $\lambda_2 = 1/2$.

(c) Since CC^T is nonsingular, the first equation can be multiplied by C and we obtain $\lambda = (CC^T)^{-1}C(A^Ta - I\hat{x})$. Inserting this expression in the first equation once again and reordering yields

$$(I - C^T(CC^T)^{-1}C)\hat{x} = (I - C^T(CC^T)^{-1}C)A^Ta.$$

Multiplying the second equation by $C^T(CC^T)^{-1}$ yields

$$C(CC^T)^{-1}C\hat{x} = C^T(CC^T)^{-1}c.$$

Adding both equations gives the requires result

$$P = (I - C^T(CC^T)^{-1}C)A^T, Q = C^T(CC^T)^{-1}.$$