Written Examination: Mathematical Models, Analysis and Simulation, Part I

Stockholm, March 27, 2009

Closed book examination 4 hours. A sum of credits of 40, homework included, is certainly enough to pass. The results will be announced no later than April 15, 2009.

Read all the questions first. They are not necessarily sorted in order of ascending difficulty. Work the easy ones first. Good luck!

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1. The defining property of the delta functional $\delta(x)$ is

$$\int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0)$$

for every smooth function $g(x)$.

   (a) How does this give "area = 1" under $\delta(x)$? What is $\int \delta(x-3)g(x)dx$, and why? **(3)**

   (b) The functional δ is the weak limit of very high, very thin square waves SW:

$$SW(x) = \begin{cases} \frac{1}{2h}, & \text{for } |x| \le h, \\ 0, & \text{for } x > h, \end{cases} \quad \text{has} \int_{-\infty}^{\infty} SW(x) g(x) dx \to g(0) \text{ as } h \to 0.$$

   For a constant $g(x) = 1$ and every $g(x) = x^n$, show that $\int SW(x)g(x)dx \to g(0)$. **(3)**

   (c) The derivative of $\delta(x)$ is the *doublet* $\delta'(x)$. Integrate by parts to compute

$$\int_{-\infty}^{\infty} g(x)\delta'(x)dx = -\int_{-\infty}^{\infty} (?)\delta(x)dx = (??) \text{ for smooth } g(x).$$

(2)

(d) The cubic splice $C(x)$ solves the fourth-order equation $u'''' = \delta(x)$. What is the complete solution $u(x)$ with four arbitrary constants? Choose those constants such that $u(1) = u''(1) = u(-1) = u''(-1) = 0$! This gives the bending of a uniform simply supported beam under a point load. Hint: What is the integral of $\delta(x)$? **(3)**

2. A predator-prey model is

$$\frac{dx}{dt} = x(1-2y)$$
$$\frac{dy}{dt} = y(x-1)$$

(a) Which is the predator and which is the prey? **(2)**

(b) Show that $D = \{(x,y)|x \geq 0, y \geq 0\}$ is a positively invariant set, i.e., no solution trajectory can leave D. **(2)**

(c) What is/are the critical point(s)? What is the type of phase portrait close to the strictly positive critical point (x^*, y^*)? **(3)**

(d) Compute approximately the period of solutions starting close to (x^*, y^*). **(2)**

3. The initial-boundary value problem

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x) = x(1-x), \quad u(0,t) = 0, \quad u(1,t) = 1$$

is discretized by the method of lines using the finite element method with the standard piecwise linear finite elements of length h. The nodes are given by

$$h = 1/N, \quad x_j = jh.$$

The basis functions are denoted by φ_j such that $\varphi_j(x_i) = \delta_{ij}$. The discrete solution is

$$u_h(x,t) = \sum_{j=1}^{N-1} u_j(t)\varphi_j(x) + a\varphi_0(x) + b\varphi_N(x).$$

We obtain the linear system of ordinary differential equations

$$\mathbf{M}\mathbf{u}_t - \mathbf{A}\mathbf{u} = \mathbf{b}, \quad \mathbf{u} = (u_1, \ldots, u_{N-1})^T.$$

(a) What are the values of a and b? What are the initial values $\mathbf{u}(0)$? Give formulas with $f(x)$ defined above. **(3)**

(b) Give formulas for the elements of $\mathbf{A}$ and $\mathbf{M}$ in terms of the basis functions. **(3)**

(c) Give formulas for the elements of the matrix **M** and the vector **b**. (i.e., evaluate the integrals appearing in (b).) **(4)**

4. For the skew-symmetric "cross product equation"

$$u' = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = Au,$$

(a) write out u'_1, u'_2, u'_3 and confirm that $u'_1 u_1 + u'_2 u_2 + u'_3 u_3 = 0$. **(2)**

(b) show that the energy $E = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2)$ is constant **(2)**

(c) find the eigenvalues of A **(3)**

(d) Show that the matrix exponential $U(t) = \exp(At)$ is unitary, i.e. $U^T U = I$, for real t. **(3)**