Final Exam
Mathematical Models, Analysis and Simulation. DN2266. Fall 2010.
Thursday, Jan 13, 2011. 14-19.

Name: ______________________________

Pers. No: ______________________________

Remember to show your work, motivate your answers well, and to clearly state the final answer. Partial credit will be given to partial solutions.

No books, notes or calculators are allowed. Good luck!

Problem 1:	________	(5)
Problem 2:	________	(7)
Problem 3:	________	(6)
Problem 4:	________	(6)
Problem 5:	________	(8)
Problem 6:	________	(4)
Problem 7:	________	(4)
Total:	________	(40)

1. For each statement below, mark if it is true or false. (No motivation required). For a statement to be true, it needs to be true for all cases. Read each statement carefully.

a) If $\mathbf{A}$ is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that $\mathbf{Ax} = \mathbf{y}$ does <u>not</u> have a solution, then there exists no vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $\mathbf{Ax} = \mathbf{z}$ has a unique solution.

b) The eigenvalues of a real symmetric matrix are always real.

c) If an $n \times n$ matrix $\mathbf{A}$ is diagonalizable, then $\mathbf{A}$ must be invertible.

d) Let $\mathbf{A}$ be an $m \times n$ matrix. If $m < n$ then $\dim(\ker(\mathbf{A})) > 0$ (i.e the dimension of the null space of $\mathbf{A} > 0$).

e) If the columns of $\mathbf{A}$ are linearly independent, then all the singular values of $\mathbf{A}$ are positive.

2. Singular value decomposition (SVD).

a) $\mathbf{A}$ is $m \times n$, $m \geq n$. Show that $\mathbf{A}$ and $\mathbf{A}^T\mathbf{A}$ has the same null space. Show that this implies that they also have the same rank.

b) What is a singular value decomposition (SVD) of a matrix $\mathbf{A}$? When does it exist? How are the singular values related to the eigenvalues of a matrix? (Which matrix?) How can the singular values tell us the rank of $\mathbf{A}$?

c) Let $\mathbf{A}$ be an $m \times n$ matrix $(m \geq n)$ with the SVD

$$\mathbf{A}_{m\times n} = \mathbf{U}_{m\times n}\mathbf{\Sigma}_{n\times n}\mathbf{V}^T_{n\times n}.$$

Assume that $\mathbf{A}$ has full rank. Show that the solution to the least squares problem $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ is

$$\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b}.$$

If $\mathbf{A}$ does not have full rank, $\mathbf{\Sigma}^{-1}$ does not exist. The pseudo-inverse of $\mathbf{A}$ is defined as $\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^T$. How is $\mathbf{\Sigma}^+$ defined?

3. Consider the 2×2 system

$$\frac{d\mathbf{u}}{dt} = \mathbf{Au}. \quad (*)$$

Assume that the general solution is given by

$$\mathbf{u}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

For each set of $\lambda_1, \lambda_2, \mathbf{v}_1, \mathbf{v}_2$ given below: Draw the phase portrait, decide what type of critical point the origin is, and whether it is stable or unstable. When applicable, include the named manifolds in the phase portrait and compute also

$$\lim_{t\to\infty} \frac{u_2(t)}{u_1(t)},$$

where $\mathbf{u}(t) = (u_1(t), u_2(t))$.

a) $\lambda_1 = -1$, $\lambda_2 = 5$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

b) $\lambda_1 = -2$, $\lambda_2 = -1/5$, $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

c) $\lambda_1 = -1 - i$, $\lambda_2 = -1 + i$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

d) Now assume that (*) has a solution of the form

$$\mathbf{u}(t) = \left[C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1/2 \\ t \end{pmatrix} \right] e^{-t}$$

What are the eigenvalues and eigenvectors of $\mathbf{A}$? Find the limit

$$\lim_{t \to \infty} \frac{u_2(t)}{u_1(t)},$$

and draw the phase portrait.

4. Consider the predator-prey model

$$\frac{du}{dt} = u(a - bv)$$
$$\frac{dv}{dt} = v(cu - d)$$

where $a, b, c, d > 0$.

a) Which is the predator and which is the prey?

b) What are the critical points? Analyze the linearized equations around the strictly positive critical point. Using this result, what can you conclude for the stability of the critical point?

5. Consider

$$\begin{aligned} u_t + au_x &= 0 \qquad x \in [0, 1], t > 0 \\ u(x, 0) &= g(x) \\ u(0, t) &= u(1, t) \quad t > 0, \end{aligned} \qquad (*)$$

where $a > 0$.

a) Which conditions do we need to check to show well-posedness for a PDE? Do it for (*).

b) What are the *characteristics* or *characteristic lines*?

c) Introduce $u_j^n = u(x_j, t_n)$, $x_j = j\Delta x$, $\Delta x = 1/N$, $j = 0, \ldots, N$ and $t_n = n\Delta t$, $n = 0, 1, \ldots$.
Consider the scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0.$$

Using von Neumann analysis, show that this scheme is always *unstable* for $\Delta t > 0$. Explain why this is the case and suggest a better scheme.

d) State Lax equivalence theorem. Why is it useful?

6. The Legendre polynomials are defined on the interval $[-1, 1]$. Let $P_n(x)$, $n = 0, 1, \ldots$ denote the Legendre polynomial of degree n. The Legendre polynomials obey the following orthogonality relation:

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0 & m \neq n, \\ \frac{2}{2n+1} & m = n. \end{cases}$$

Give the expansion of a function $f(x)$ in the Legendre polynomials and give the formula for the coefficients in the expansion.

7. Consider the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = d\frac{\partial^3 u}{\partial x^3}$$

defined on $x \in [0, L]$ with boundary conditions $u(0, t) = u(L, t)$, where c and d are constants.

Let $u^N(x, t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t) e^{\frac{2\pi i k x}{L}}$ be a spectral expansion of u.

Derive the Galerkin equations for the expansion coefficients $\hat{u}_k(t)$.