Final Exam (Re-exam)
Mathematical Models, Analysis and Simulation. DN2266.
Instructor: Anna-Karin Tornberg
Saturday, Feb 18, 2012. 9-14.

Remember to show your work, motivate your answers well, and to clearly state the final answer. Partial credit will be given to partial solutions.

1. (4p). For each statement below, mark if it is true or false. (No motivation required). For a statement to be true, it needs to be true for all cases. Read each statement carefully.

a) For any $n \times n$ matrix $\mathbf{A}$, the sum of the eigenvalues equals the sum of the diagonal elements.

b) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. The eigenvalues of $\mathbf{A} + \mathbf{B}$ can be determined from the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.

c) Let $\mathbf{A}$ be any real $m \times n$ matrix. Then $\mathbf{A}^T\mathbf{A}$ is symmetric positive definite.

d) Any symmetric $n \times n$ matrix $\mathbf{A}$ is diagonalizable.

2. (8p). Singular value decomposition.
Let $\mathbf{A}$ be an $m \times n$ matrix $(m \geq n)$ with the SVD

$$\mathbf{A}_{m\times n} = \mathbf{U}_{m\times n}\mathbf{\Sigma}_{n\times n}\mathbf{V}^T_{n\times n}.$$

a) When does an SVD exist? What can you say about the matrices $\mathbf{U}$, $\mathbf{\Sigma}$ and $\mathbf{V}$? When are all the singular values of $\mathbf{A}$ strictly positive?

b) Given the singular values of $\mathbf{A}$, σ_i, $i = 1, \ldots, n$, show that the eigenvalues of the matrix $\mathbf{A}^T\mathbf{A}$ are σ_i^2.
Hint: Compute $\mathbf{A}^T\mathbf{A}$. What is the eigenvalue decomposition of a matrix?

c) What do we need to assume for $\mathbf{A}$ to know that $(\mathbf{A}^T\mathbf{A})^{-1}$ exists? Express $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ and $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ in terms of $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$.

d) If $\mathbf{A}$ has full rank, show that the solution of $\min_{\mathbf{x}} ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2$ is $\mathbf{x} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T\mathbf{b}$.
Hint: What equation is $\mathbf{x}$ the solution of?

3. (8p). Dynamical systems.

a) Consider the first order differential equation

$$\frac{dx}{dt} = x - x^3.$$

Find the critical points and determine if they are stable or unstable. Sketch the trajectories in the (t, x)-plane of the solutions with initial conditions $x(0) = -2, -\epsilon, \epsilon, 2$, where $0 < \epsilon \ll 1$.

b) Consider the second order differential equation

$$\frac{d^2x}{dt^2} = x - x^3.$$

Find the critical points and determine if they are stable or unstable. Sketch the phase plane, covering all critical points.
Note: The linear stability analysis does in this case determine the behavior of this nonlinear problem around the critical points.

4. (8p). PDEs.

a) Consider the following equations

$$u_t + au_x = 0, \tag{1}$$

$$u_t = cu_{xx}, \quad c > 0, \tag{2}$$

on $-\infty < x < \infty$, $t \geq 0$.
For each of the equations above, insert the ansatz

$$u(x,t) = \hat{u}e^{i(\omega t + \beta x)},$$

and find the relation between ω and β, i.e. express ω in terms of β. Comment on the properties of the solutions.

b) Introduce $u_j^n = u(x_j, t_n)$, $x_j = j\Delta x$, $t_n = n\Delta t$, $\Delta x, \Delta t > 0$. The finite difference scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \quad (a > 0). \tag{3}$$

approximates (1) to first order in Δx and Δt. By Taylor expansion, find the PDE for which this discretization is a second order approximation. Make sure that the additional term in the PDE contains only x derivatives. Using this, comment on the errors in (3) in approximating (1).

5. (8p). Fisher's equation is a reaction/diffusion equation which was originally proposed as a model for the spatial and temporal propagation of a virile gene in an infinite medium. It is a one-dimensional reaction diffusion model for the evolution of the concentration of the infected population, $u(x', t')$, with a quadratic reactive term corresponding to logistic growth. The equation is defined by

$$\frac{\partial u}{\partial t'} = D\frac{\partial^2 u}{\partial x'^2} + ku(1-u), \quad -\infty < x' < \infty, \quad t' \geq 0,$$

where D is a diffusion coefficient, and k is a reaction rate coefficient.

a) Show that by a proper choice of length and time scales, L and T, ($x' = Lx$ and $t' = Tt$), the equation becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u),$$

What are L and T?

b) Consider the equation above with periodic boundary conditions on $x \in [0, 1]$, and periodic initial conditions. Let $u^N(x,t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t) e^{\frac{2\pi i k x}{L}}$ be a spectral expansion of u.

i) Derive the Galerkin equations for the expansion coefficients $\hat{u}_k(t)$.

ii) Introduce a simple time-stepping scheme and describe a pseudo-spectral algorithm (i.e. without removing aliasing errors).

6. (4p). The Chebyshev polynomials are defined on the interval $[-1, 1]$. Let $T_n(x)$, $n = 0, 1, \ldots$ denote the Chebyshev polynomial of degree n. The Chebyshev polynomials obey the following orthogonality relation:

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{1}{\sqrt{1-x^2}}\, dx = \begin{cases} 0 & m \neq n, \\ \pi & m = n = 0, \\ \frac{\pi}{2} & m = n \neq 0. \end{cases}$$

Give the expansion of a function $f(x)$ in the Chebyshev polynomials and give the formula for the coefficients in the expansion.