Lecture 2:
Linear Algebra: Eigenvalues, systems of differential equations, etc., S. Ch 1

1 Main problems of Numerical Linear Algebra

Find eigenvector(s) **x** and complex eigenvalue(s) λ
"Standard" eigenvalue problem:

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Generalized eigenvalue problem:

$$\mathbf{Ax} = \lambda\mathbf{Bx}$$

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Matlab: lam = eig(A);
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Examples

1.1 The Markov chain equilibrium distribution - Lect 1.

Uncommon that

- the eigenvalue is known
- the eigenvalue with largest absolute value is wanted

Here is a plot of a computational experiment on the distribution of complex eigenvalues to 600 40x40 Markov random matrices, i.e., a 2D histogram of 600 x 40 = 2400 points.

1. the 600 peak at the extreme right (not at the right spot ...)
2. flat distribution around 0 with a radius of ≈ 0.1
3. a ridge of real eigenvalues

I think 1. is easy and can think of a reason for 3., but 2. ?? One would think that since **A** has only positive entries the eigenvalues would flock to the right hand plane. *Not so*. You may want to check on Alan Edelman's lectures on random matrices.
While on the subject of guessing eigenvalues, look at the Gershgorin circle theorem (p. 570):
Every eigenvalue is in the union of circles C_i, $i = 1,2,\ldots,n$

$$C_i : |\lambda - a_{ii}| \le \sum_{j \ne i} |a_{ij}|$$

The G.-circles for a Markov matrix are all centered on the interval [0,1] and pass through 1, so all contained in the unit circle. But the probable eigenvalues occupy but a minuscule portion of it.

2 Quadratic forms $q(\mathbf{x}) = \mathbf{x}^T\mathbf{Kx}$

Note: Uses only the symmetric part $\frac{1}{2}(\mathbf{K}+\mathbf{K}^T)$ so consider **K** symmetric. We will measure the size of **x** by its Euclidean norm,

$$\|\mathbf{x}\|_2 = \left(\sum |x_k|^2\right)^{1/2} = \sqrt{\mathbf{x}^T\mathbf{x}}$$

and remind you of the triangle inequality, the multiplication by scalar, the Cauchy-Schwarz inequality

$$\mathbf{x}^T\mathbf{y} \le \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2$$

and the definition of the operator norm of a linear operator (matrix!) $\|\mathbf{A}\|_2$, induced by the vector norm

$$\|\mathbf{A}\|_2 = \max \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2} = \max \sqrt{\frac{\mathbf{x}^T \mathbf{A}^T \mathbf{Ax}}{\mathbf{x}^T \mathbf{x}}}$$

and the Rayleigh quotient (p 219)

$$R_{\mathbf{K}}(\mathbf{x}) = \mathbf{x}^T\mathbf{Kx}/\mathbf{x}^T\mathbf{x}$$

The norm shows the maximal magnification possible in the mapping. Let us compute it by finding the max. of the Rayleigh quotient by differentiation:

$$\frac{\partial}{\partial x_k}\left(\mathbf{x}^T\mathbf{Kx}\right) = \frac{\partial}{\partial x_k}\sum_{i,j} k_{ij}x_i x_j = \sum_{i,j} k_{ij}\left(\underbrace{\frac{\partial x_i}{\partial x_k}}_{\delta_{ik}} x_j + x_i \underbrace{\frac{\partial x_j}{\partial x_k}}_{\delta_{jk}}\right) =$$

$$= \sum_j k_{kj}x_j + \sum_i k_{ik}x_i = 2(\mathbf{Kx})_k$$

so $\partial R/\partial x_k = 0$, $k = 1,\ldots,n$, if (and only if)

$$2\mathbf{Kx}\cdot\mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{Kx}\cdot 2\mathbf{x} = 0 : \mathbf{Kx} = \underbrace{R_{\mathbf{K}}(\mathbf{x})}_{\lambda}\mathbf{x}$$

Theorem:
If $\mathbf{Kx^*} = \lambda\mathbf{x^*}$, then $\mathbf{x^*}$ is a stationary point of $R_{\mathbf{K}}(\mathbf{x})$ and $\lambda = R_{\mathbf{K}}(\mathbf{x^*})$, and conversely.

It follows that

$$\|\mathbf{A}\|_2 = \sqrt{\text{largest eigenvalue of } \mathbf{A}^T\mathbf{A}}$$

Note: minimization of quadratic forms *with a single quadratic constraint* also leads to eigenvalue problems.

3 Linear differential equations with constant coefficients

(S p 53, Ch 2.1, 2.2)

Basics:

Exponentials are eigenfunctions of differential and difference operators with constant coefficients. Usually the independent variable is now time t.

Let $D = d/dt$. Then $D\exp(\lambda t) = \lambda\exp(\lambda t)$, and

$$p(D)e^{\lambda t} = \sum_{k=0}^{n} a_k D^k\left(e^{\lambda t}\right) = \sum_{k=0}^{n} a_k \lambda^k\left(e^{\lambda t}\right) = p(\lambda)e^{\lambda t}$$

The analogue holds for difference operators (p 54 ff), e.g.

$$\Delta u(t_k) = u(t_{k+1}) - u(t_k),\ t_{k+1} - t_k = h,\ k = \ldots,-1,0,1,\ldots$$

Then

$$\Delta e^{\lambda t} = e^{\lambda(t+h)} - e^{\lambda t} = \mu e^{\lambda t}, \mu = e^{\lambda h} - 1$$

Note that

$$\lim_{h\to 0} \mu/h = \lambda$$

So

$$p(\Delta)e^{\lambda t} = \sum_{k=0}^{n} a_k \Delta^k\left(e^{\lambda t}\right) = \sum_{k=0}^{n} a_k \mu^k\left(e^{\lambda t}\right) = p(\mu)e^{\lambda t}$$

3.1 Examples with complex eigenvalues: Rigid body rotation

3.1.1 In a plane

The velocity at (x,y) of rotation with angular velocity ω around the origin is

$(dx/dt, dy/dt) = \omega(-y,x)$

or

$$\frac{d}{dt}\mathbf{u} = \underbrace{\omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{A}}\mathbf{u}, \mathbf{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Look for special solution vectors **u** = exp(λ*t*)**v**, **v** some constant vector:

$$\frac{d}{dt}\mathbf{u} = \lambda e^{\lambda t}\mathbf{v} = \mathbf{Au} = \mathbf{A}e^{\lambda t}\mathbf{v} \Leftrightarrow \lambda\mathbf{v} = \mathbf{Av}$$

The eigenvalues of **A** are imaginary, +/–ω *i* and the eigenvectors are $(-1,i)^T$ and $(1,i)^T$ so any linear combination

$$\mathbf{u}(t) = a\begin{pmatrix} -1 \\ i \end{pmatrix}e^{+i\omega t} + b\begin{pmatrix} 1 \\ i \end{pmatrix}e^{-i\omega t}$$

satisfies the equation. There are enough (2) integration constants to satisfy initial conditions like $\mathbf{u}(0) = (x_0,y_0)$, so this is the general solution. It looks complex, but real initial conditions fix that:

$$\mathbf{u}(t) = \frac{-x_0 - iy_0}{2}\begin{pmatrix} -1 \\ i \end{pmatrix}e^{+i\omega t} + \frac{x_0 - iy_0}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}e^{-i\omega t} = \begin{pmatrix} x_0 \cdot \cos\omega t - y_0 \cdot \sin\omega t \\ x_0 \cdot \sin\omega t + y_0 \cdot \cos\omega t \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}}_{\text{Rotates angle } \omega t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The manipulations become trivial if we use complex variables: $z(t) = x(t) + iy(t)$. Then

$dz/dt = i\omega z$ and $z(t) = \exp(i\omega t)z(0)$

so

$\text{angle}(z(t)) = \text{angle}(z(0)) + \omega t$, $|z(t)|=|z(0)|$

3.1.2 In 3-space

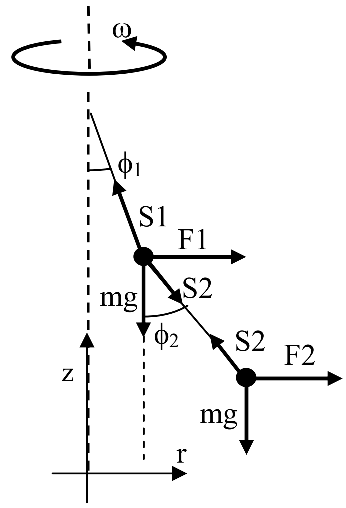
You may want to think about the 3D counterpart, rotation with angular velocity ω around a unit length vector (w_1,w_2,w_3)

$$\frac{d}{dt}\mathbf{u}(t) = \omega\mathbf{w} \times \mathbf{u} = \mathbf{Au}, \mathbf{A} = \omega\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

Compute the eigenvalues of **A**! It is anti-symmetric. Hint: One is zero ... compute det**A** to see this. Or find by inspection a non-zero vector **p** such that **Ap** = 0.

4 Bifurcation ... linearized

(see also S. p 108-109)
Consider a double pendulum with two arms of lengths *l*. Its motion is constrained by the joints to a plane, rotating around a vertical axis with angular velocity ω. Compute its equilibrium position, assuming it has small angles ϕ1, ϕ2!
The equilibrium equations for the mass points 1 and 2 are



$$F1 = ml\omega^2 \sin\phi_1, F2 = ml\omega^2(\sin\phi_1 + \sin\phi_2)$$

$$1,r: ml\omega^2 \sin\phi_1 + S_2 \sin\phi_2 - S_1 \sin\phi_1 = 0$$

$$1,z: S_1 \cos\phi_1 - S_2 \cos\phi_2 - mg = 0$$

$$2,r: ml\omega^2(\sin\phi_1 + \sin\phi_2) - S_2 \sin\phi_2 = 0$$

$$2,z: S_2 \cos\phi_2 - mg = 0$$

Eliminate the forces S_i

$$\left.\begin{matrix} S_1 = 2mg/\cos\phi_1 \\ S_2 = mg/\cos\phi_2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} (\sin\phi_1 + \sin\phi_2)\lambda = \tan\phi_2 \\ \lambda \sin\phi_1 = 2\tan\phi_1 - \tan\phi_2 \end{matrix}\right., \lambda = \frac{l\omega^2}{g}$$

Approximate the trig-functions to produce the final linear system for *small* angles:

$$\left\{\begin{matrix} (\phi_1 + \phi_2)\lambda = \phi_2 \\ \lambda\phi_1 = 2\phi_1 - \phi_2 \end{matrix}\right. \Rightarrow \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

which shows that: unless λ is an eigenvalue, there is only the trivial solution: the pendulum hangs straight down. For sufficiently small ω, $\lambda < 2 - \sqrt{2}$ this is the case, and at this λ-value the solution suddenly becomes a multiple of the first eigenvector. The multiple is not determined by the linear approximate model; The two eigenvectors are

$$\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \lambda = 2 - \sqrt{2}, \text{ and } \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \lambda = 2 + \sqrt{2},$$

Both shapes can be provoked when you twirl a hanging rope fast enough. What actually happens when λ is increased past $2 - \sqrt{2}$? The story requires that we consider the full time-dependent non-linear dynamic system problem, later.