Lecture 2 addendum

1 The prize problem:

Compute

$$\det(\mathbf{I} + \mathbf{u}\mathbf{u}^T) = 1 + \mathbf{u}^T\mathbf{u}$$

without calculating the eigenvalues. A solution was provided by Marwa, who observed:

$$\det(\mathbf{I} + \mathbf{u}\mathbf{u}^T) = \left|\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline -\mathbf{u} & \mathbf{I} \end{array}\right|$$

and the latter matrix is much better because it is so sparse. The next step uses a block-factorization. The standard Gaussian elimination eliminates the elements in the –**u** column to produce … what Marwa observed,

$$\left(\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline -\mathbf{u} & \mathbf{I} \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline -\mathbf{u} & \mathbf{I} \end{array}\right)\left(\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline 0 & \mathbf{I} + \mathbf{u}\mathbf{u}^T \end{array}\right)$$

but we may instead eliminate the $\mathbf{u}^T$-row,

$$\left(\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline -\mathbf{u} & \mathbf{I} \end{array}\right) = \left(\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline 0 & \mathbf{I} \end{array}\right)\left(\begin{array}{c|c} 1 + \mathbf{u}^T\mathbf{u} & 0 \\ \hline -\mathbf{u} & \mathbf{I} \end{array}\right)$$

which you can see by doing the elimination starting at the lower right corner, working upward. Taking the determinants of the two right hand side matrices produces the desired formula.

Slightly more general block-elimination formulas, assuming that the inverses exist are

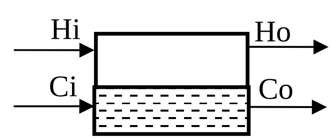
$$\left(\begin{array}{c|c} \mathbf{A}_{pxp} & \mathbf{B}_{pxr} \\ \hline \mathbf{C}_{rxp} & \mathbf{D}_{rxr} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I_p} & \mathbf{0} \\ \hline \mathbf{CA}^{-1} & \mathbf{I_r} \end{array}\right)\left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{0} & \mathbf{D} - \mathbf{CA}^{-1}\mathbf{B} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{I_p} & \mathbf{BD}^{-1} \\ \hline \mathbf{0} & \mathbf{I_r} \end{array}\right)\left(\begin{array}{c|c} \mathbf{A} - \mathbf{BD}^{-1}\mathbf{C} & \mathbf{0} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right)$$

which define the *Schur complements*

$\mathbf{D}\text{-}\mathbf{CA}^{-1}\mathbf{B}$ and $\mathbf{A}\text{-}\mathbf{BD}^{-1}\mathbf{C}$

This prize problem is now closed.

2 Example: A simple heat exchanger model



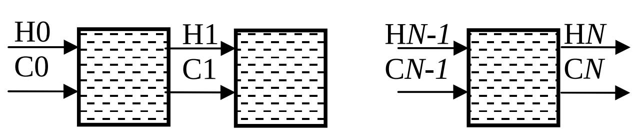
A heat exchanger unit takes in hot fluid at temp. Hi and cold at Ci and delivers the hot stream at To and the cold stream at Co: Energy conservation is expressed by

$$Ho + Co = Hi + Ci$$

and the heat transfer efficiency by

$$Ho - Co = \alpha(Hi - Ci), 0 < \alpha < 1$$

The task is to compute the output temperatures from a system of *N* units connected in series:



where

$$\begin{cases} H_{k+1} + C_{k+1} = H_k + C_k \\ H_{k+1} - C_{k+1} = \alpha(H_k - C_k) \end{cases}, k = 0,1,\ldots, N-1$$

We write this as a vector recursion for $\mathbf{y}_k = (H_k, C_k)^T$:

$$\mathbf{A}\mathbf{y}_{k+1} = \mathbf{B}\mathbf{y}_k, \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 \\ \alpha & -\alpha \end{pmatrix}:$$

$$\mathbf{y}_{k+1} = \mathbf{K}\mathbf{y}_k, \mathbf{K} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, a = \frac{1+\alpha}{2}, b = \frac{1-\alpha}{2}$$

Now use diagonalization to write the solution

$$\mathbf{y}_k = \mathbf{K}^k\mathbf{y}_0 = \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^{-1}\mathbf{y}_0$$

where **S** is the eigenvector matrix and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues of **K**

$$\mathbf{KS} = \mathbf{S\Lambda}$$

Note that **K** is symmetric so **S** is orthogonal: $\mathbf{S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The eigenvalues are 1 and α, and we obtain

$$\mathbf{y}_k = \begin{pmatrix} H_k \\ C_k \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\alpha^k & 1-\alpha^k \\ 1-\alpha^k & 1+\alpha^k \end{pmatrix}\begin{pmatrix} H_0 \\ C_0 \end{pmatrix}$$

This we could have written immediately, because the original equations are already diagonal in H+C and H-C – the eigenvectors:

$$H_k + C_k = H_0 + C_0, H_k - C_k = \alpha^k (H_0 - C_0)$$

No matter how many units are connected, the cold stream out has a temp. < (H0+C0)/2.

3 Prize problem II

Given $N = 5$, $H_0 = 100$, $C_0 = 0$, what is the max. possible output temp C_5? You are allowed to connect the pipes between the units in any way (but the cold and hot sides must be separated). What way? Draw a sketch. You'll be happy when you see the pattern.

4 The heat exchanger as dynamic system

It takes some time for the fluid to flow through a unit, say time Δt. Set time-points

$$t_n = n\Delta t, n = 0,1,..$$

and let the temperatures out of unit k at time t_n are $\mathbf{y}_k^n = \begin{pmatrix} H_k^n \\ C_k^n \end{pmatrix}$, then the dynamics is expressed by

$$\mathbf{y}_k^{n+1} = \mathbf{K}\mathbf{y}_{k-1}^n, k = 1,2,..., N$$

with the input $\mathbf{y}_0^n = \mathbf{f}$ is turned on at $t = 0$, vanishes before t = 0, so the initial conditions are $\mathbf{y}_k^0 = 0$. Putting all the **y** into a big vector **Y**, we can write – using matlab **; -** notation for putting columns over one another –

$$\mathbf{Y}^{n+1} = \tilde{\mathbf{K}}\mathbf{Y}^n + \mathbf{F}, \mathbf{Y}^n = (\mathbf{y}_1^n; \mathbf{y}_2^n; ...; \mathbf{y}_N^n), \mathbf{F} = (\mathbf{f}; 0; ...; 0)$$

$$\tilde{\mathbf{K}} = \begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ \mathrm{K} & 0 & 0 & ... & 0 \\ 0 & \mathrm{K} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \mathrm{K} & 0 \end{pmatrix}$$

Claim: The system reaches steady-state at t_N.
Proof:
We have

$$\mathbf{Y}^n = \tilde{\mathbf{K}}\mathbf{Y}^{n-1} + \mathbf{F} = \tilde{\mathbf{K}}(\tilde{\mathbf{K}}\mathbf{Y}^{n-2} + \mathbf{F}) + \mathbf{F} = \tilde{\mathbf{K}}^2\mathbf{Y}^{n-2} + (\tilde{\mathbf{K}} + \mathbf{I})\mathbf{F} =$$

$$\tilde{\mathbf{K}}^3\mathbf{Y}^{n-3} + (\tilde{\mathbf{K}}^2 + \tilde{\mathbf{K}} + \mathbf{I})\mathbf{F} = \ldots = (\tilde{\mathbf{K}}^{n-1} + \tilde{\mathbf{K}}^{n-2} + \ldots + \tilde{\mathbf{K}}^2 + \tilde{\mathbf{K}} + \mathbf{I})\mathbf{F} =$$

$$\sum_{k=0}^{n-1} \tilde{\mathbf{K}}^k \cdot \mathbf{F}$$

Now the matrix is *nilpotent*: $\tilde{\mathbf{K}}^k = 0, k = N, N+1, \ldots$. It inherits its nilpotency from the block-structure, which acts on a vector by shifting the blocks up (in the numbering, i.e., *down* in the typography) one step:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

It follows that

$$\mathbf{Y}^n = \sum_{k=0}^{n-1} \tilde{\mathbf{K}}^k \cdot \mathbf{F} = \sum_{k=0}^{\min(n-1, N)} \tilde{\mathbf{K}}^k \cdot \mathbf{F}$$

QED

What happens if **A** is not nilpotent in the recursion $\mathbf{x}^{n+1} = \mathbf{A}\mathbf{x}^n + \mathbf{f}, \mathbf{x}^0 = 0$?
IF a limit **x*** exists as n grows without bound then $\mathbf{x}^* = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}$.

Claim
If the maximum modulus $\rho(\mathbf{A})$ (the spectral radius) of any eigenvalue of **A** satisfies $\rho < 1$, the recursion converges. In this case also the matrix series converges,

$$\lim_{n \to \infty} \mathbf{A}^n = 0 \text{ and } (\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k \text{ (the Neumann series)}$$

This is easy to prove for diagonalizable matrices but slightly more complicated in general.