

Lecture 5: Phase portraits and bifurcation

The phase portraits for linear, constant coefficient 2D systems

$$du/dt = Au$$

can be completely characterized, and we proceed to make a map of the different types. The eigenvalues of A are λ_1 and λ_2 , and the corresponding eigenvectors v_1 and v_2 . The plots are shown in (x_1, x_2) -coordinates. For diagonalizable systems with real eigenvalues this means

$$u(t) = x_1(t)v_1 + x_2(t)v_2$$

For complex eigenvalues, $\lambda = \mu + i\omega$, $v_2 = \text{conj}(v_1)$ and we choose $w_+ = \text{Re}(v)$ and $w_- = \text{Im}(v)$ as basis vectors. Then $Aw_+ = \mu w_+ + \omega w_-$, $Aw_- = -\omega w_+ + \mu w_-$ and

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \mu & \omega \\ -\omega & \mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation is

$$0 = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = \lambda^2 - p\lambda + q, p = \text{tr}A, q = \det(A)$$

$$D = p^2/4 - q$$

Real: $D > 0$; Complex: $D < 0$; Double: $D = 0$.

The (p, q) - parametrization discriminates all the cases, except the two variants of $D = 0$, diagonalizable and defective (= non-diagonalizable).

1. $D > 0$

i) Negative: $\lambda_1 < \lambda_2 < 0$

$$x_1 = x_1^0 e^{\lambda_1 t}, x_2 = x_2^0 e^{\lambda_2 t}, \text{both} \rightarrow 0$$

$$x_1 / x_2 \rightarrow 0 : \text{final approach to 0 along } x_1 = 0$$

A *stable node*.

ii) Both signs: $\lambda_1 < 0 < \lambda_2$

$$x_1 = x_1^0 e^{\lambda_1 t} \rightarrow 0, x_2 = x_2^0 e^{\lambda_2 t} \rightarrow \infty$$

$$x_1 / x_2 \rightarrow 0 : \text{final divergence to } \infty \text{ along } x_1 = 0$$

A *saddle*, unstable, except for points with $x_2 = 0$

iii) Positive: $0 < \lambda_1 < \lambda_2$

$$x_1 = x_1^0 e^{\lambda_1 t}, x_2 = x_2^0 e^{\lambda_2 t}, \text{both} \rightarrow \infty$$

$$|x_2| = C|x_1|^{\lambda_2/\lambda_1}$$

Same as i), with direction on trajectories reversed, and 1 and 2 switched, an *unstable node*.

2. $D < 0$, complex eigenvalues $\mu \pm i\omega$:

i) $\mu > 0$, *unstable spiral*

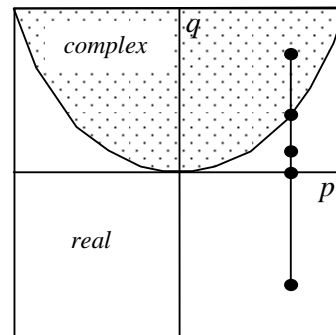
ii) $\mu = 0$, a *center* (neutrally stable)

iii) $\mu < 0$, *stable spiral*

Note: If the matrix is singular, $q = 0$, the origin is not the only critical point. Rather, all multiples of the zero eigenvector v are critical, so the dynamics takes place along lines at an angle to v – a solution stays on a line.

Exercise:

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \text{ What angle does a trajectory make with the } u_1\text{-axis?}$$



Example:

<http://www-math.mit.edu/daimp/LinPhasePorMatrix.html>

to play with the entries of the matrix, shows eigenvalues and phase portrait, and names it.

The first complete map of this kind was probably made by H.Poincaré about 100 years ago. Poincaré's (half)sphere, mapped onto the unit disk:

$$\mathbf{v} = \mathbf{u} / \sqrt{1 + u_1^2 + u_2^2}$$

shows the *whole* phase space.

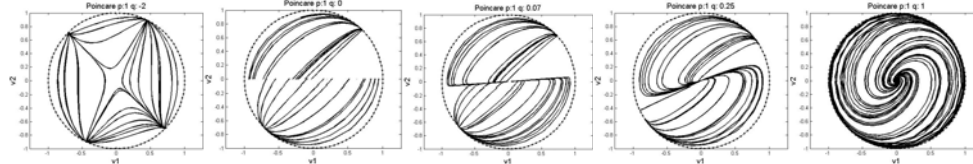
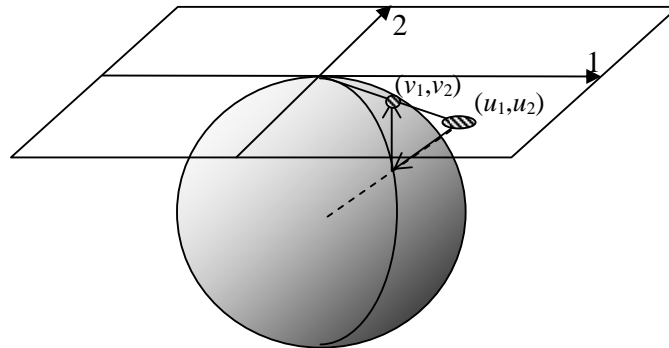
The Poincaré plots below

show a saddle, a degenerate

($q=0$) node, a proper node, a

“one-tangent” degenerate node,

and a spiral (focus). The plots were made by integration forward *and* backward in time, so both stable and unstable manifolds are seen.



The portraits correspond to the points marked on the (p,q)-map above, with “bifurcations” - changes to the type of portrait - at $q = 0$ and 0.25 .

Bifurcations

Example

A point mass m on a rod of length l rotates around the z -axis with angular velocity ω .

Gravitational acceleration is g in the negative z -direction. No friction or other damping. We derive the dynamics by Lagrange's equations, and in keeping with Lagrange's tradition there is no drawing:

Gravitational + “centrifugal” potential energy:

$$W = mgz + 1/2m(l\sin\theta)^2\omega^2 = -mgl\cos\theta + 1/2m\omega^2 l^2 \sin^2\theta,$$

Kinetic energy:

$$T = 1/2ml^2(d\theta/dt)^2$$

Equations of motion:

$$\frac{d}{dt} \frac{\partial(T-W)}{\partial \dot{\theta}} = \frac{\partial(T-W)}{\partial \theta} : ml^2\ddot{\theta} = -mgl \sin\theta + m\omega^2 l^2 \sin\theta \cos\theta,$$

$$\ddot{\theta} = \sin\theta(\lambda \cos\theta - 1), \lambda = \frac{\omega^2 l}{g}$$

where the time units were chosen to match the natural frequency of small oscillations.

The state vector is $\mathbf{u} = (\theta, d\theta/dt)^T$. The critical points are $d\theta/dt = 0$,

$$\sin(\theta) = 0: \theta = k\pi, k = 0, +1, +2, \dots,$$

$$\text{and if } \lambda > 1, \text{ also } \theta = \arccos(1/\lambda).$$

Stability is determined by the second derivative of W ,

$$\frac{d^2W}{d\theta^2} = \lambda \cos 2\theta - \cos \theta :$$

$\theta = 0 : \lambda - 1$, all $\lambda < 0$, neutrally stable, for $\lambda < 1; > 0$, unstable, for $\lambda > 1$

$\theta = \pm\pi : \lambda + 1$, all $\lambda > 0$, unstable

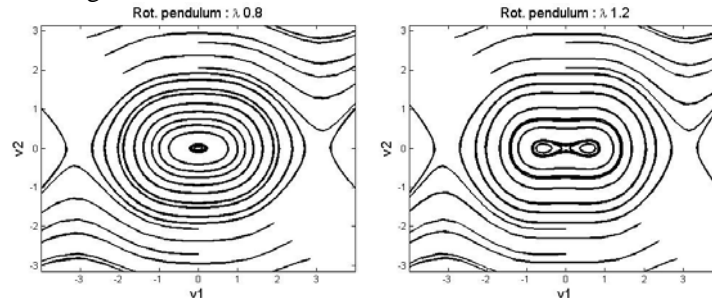
$$\lambda \cos \theta = 1 : \frac{1 - \lambda^2}{\lambda}, \lambda \geq 1. < 0, \text{ neutrally stable.}$$

Phase portraits

Note that the θ -axis is periodic, so $+\pi$ and $-\pi$ are identical.

Left, $\lambda = 0.8$: the origin is a center, saddles at $\pm\pi$.

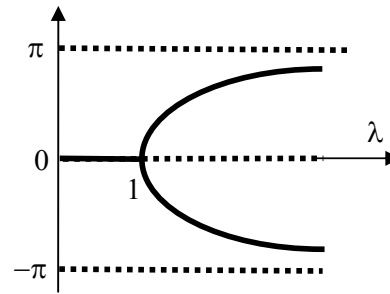
Right, $\lambda = 1.2$: the origin is a saddle, $\pm 33.6^\circ$ are centers, saddles at $\pm\pi$



Exercise:

Check the claims about the phase portraits by computing the Jacobian and its p and q !

The equilibria are continuous functions of λ , as shown in the *bifurcation* diagram of the roots (critical points, equilibria, ...), right. Dotted: unstable, continuous line: center. A *pitchfork bifurcation*.



At the bifurcation, the equilibrium depends very sensitively on λ .

$$\cos \theta = 1 - \theta^2/2 + \dots, \lambda = 1 + \delta:$$

$$\theta = \sqrt{2\delta} + \dots$$

so the root is *not* differentiable as function of the parameter λ there.

Hopf bifurcation

The last example is the famous Hopf bifurcation: at the critical parameter value the root becomes unstable, but there is no other (finite) critical point, and the solution becomes a *limit cycle*, approaching a periodic solution. The screaming noise emitted by a PA-system when the microphone is brought close to the loudspeaker is a dynamical system of this kind. We refrain from more concrete modelling and stay with the abstract, noting that the equation admits a very simple complex form, $z = x + iy$,

$$\dot{z} = z(\lambda + (\mu + i\omega)|z|^2)$$

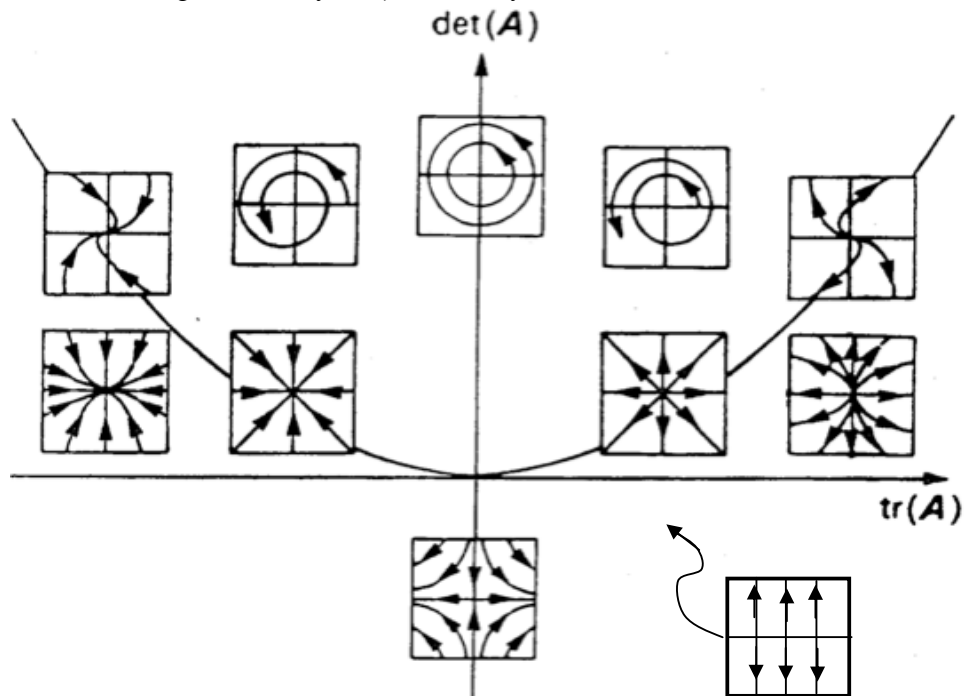
$$\dot{x} = \lambda x + (x^2 + y^2)(\mu x - \omega y)$$

$$\dot{y} = \lambda y + (x^2 + y^2)(\omega x + \mu y)$$

The origin is an (isolated) critical point with Jacobian $\lambda \mathbf{I}$, a *star*, stable if $\lambda < 0$, unstable if $\lambda > 0$, and no other, if ω is non-zero, as can be seen e.g. by writing the equations in polar coordinates (r, ϕ) :

$$\frac{1}{2} \frac{d}{dt} r^2 = (\lambda + r^2 \mu) r^2, \dot{\phi} = \omega r^2$$

So if $\mu < 0$, $\lambda > 0$ the r -equation has a positive stable critical point (like the logistic equation), $r \rightarrow \sqrt{\frac{\lambda}{-\mu}}$, but ϕ increases monotonically. The trajectory approaches circular motion with angular velocity $\omega\lambda/|\mu|$: a limit cycle.



The last picture is the artist's impression of the possible cases, copied from lecture notes C.Trygger, unknown source