Lecture 5, addendum: Crystal growth and supersonic focusing.

Crystal growth

A crystal grows in a super-saturated solution so that:
the amount of material solidified per unit time is proportional to the exposed area:

$$dV/dt = kA, V(0) = 0$$

and the crystal growth starts at $t = 0$. This model is not "closed" because the relation between volume and surface area depends on the shape. For a sphere, $A = (36\pi)^{1/3}V^{2/3}$, so

$$dV/dt = cV^{2/3} \qquad (*)$$

and this right hand side is NOT Lipschitz at $V = 0$, so there are multiple solutions, for instance

$$V = 0 \text{ and } V(t) = (ct/3)^3$$

As soon as $V > 0$, the rest is determined, but the model does not show how the growth begins. The physicist will mention the "nucleation" process, which relies on the existence of "seeds" – dust particles, etc.
However, before we get too excited about the precision of the simple model, we may express it in terms of the radius to obtain

$$dr/dt = c_2, r(0) = 0$$

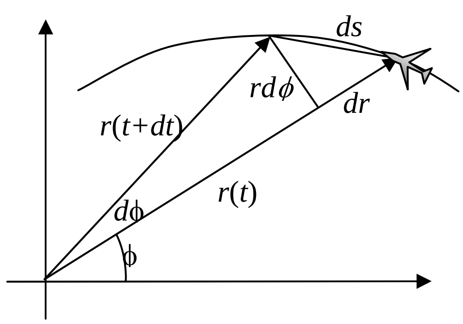
which starts growth at $t = 0$. Question: Where did all the other solutions to (*) go?

Supersonic focusing

Is it possible to fly a speeding jet so *all* sound emitted along the path hits a given point simultaneously? Focusing of waves from a speedboat can be observed when it turns a tight circle, so something similar should be possible.
We make a simple model using Huygen's principle:

> "Every point on a wave front is the center of a spherical wave; the wavefront at the next instant in time becomes the envelope of all these small spherical waves."

Huygens formulated this in the seventeenth century, a remarkable feat, since there was no notion of "wave" partial differential equations at the time. Instead, the reasoning was done with a particle-type model for sound or light propagation.
Here is the picture of the plane flying at speed v:



and we conclude

$$t + dt + \frac{r(t+dt)}{c} = t + \frac{r(t)}{c} \Rightarrow \frac{dr}{dt} = -c$$

where c is the speed of sound. The other relation is the arclength element and the flight speed,

$$ds^2 = dr^2 + (rd\phi)^2,$$

$$v = \frac{ds}{dt} = \sqrt{(dr/dt)^2 + r^2(d\phi/dt)^2}$$

which gives the angular rate

$$\frac{d\phi}{dt} = \frac{\sqrt{v^2 - c^2}}{r}, \text{ for } v > c$$

The plane must be supersonic for this to work. The easy way to find the flight path – *in the phase plane* – is to combine the r and ϕ equations to

$$\frac{dr}{d\phi} = -\frac{r}{\sqrt{M^2-1}}, M = v/c: \quad \ln\frac{r}{r(\phi_0)} = -\frac{\phi - \phi_0}{\sqrt{M^2-1}}$$

The plane must fly a *logarithmic spiral*.