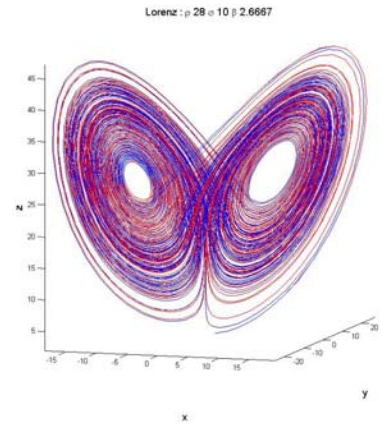
Lecture 6, addendum: The Lorenz attractor, the van der Pol equation, and the Poincaré-Bendixson theorem

Let us look at phase space for a 3-system, a very famous one, which publicized concepts like "the butterfly effect" – I think James Gleick coined the phrase in *Chaos* and "chaos" and "strange attractors":

$$\begin{cases} \dfrac{dx}{dt} = \sigma(y-x) \\ \dfrac{dy}{dt} = x(\rho - z) - y \\ \dfrac{dz}{dt} = xy - \beta z \end{cases}$$



This is a hard-handedly simplified model of weather: convection and heating and the spinning of the earth. σ is called the Prandtl number (balance between thermal conduction and viscosity), and ρ is called the Rayleigh number (buoyancy and convection).
Usually we take $\sigma = 10$ and $\beta = 8/3$ and vary ρ. The red and blue trajectories start 10^{-4} apart; but after a while they drift apart, and flip from one leaf of the structure to the other, seemingly at random. But this is not a numerical artefact; the system is very sensitive to perturbations – the butterfly effect.

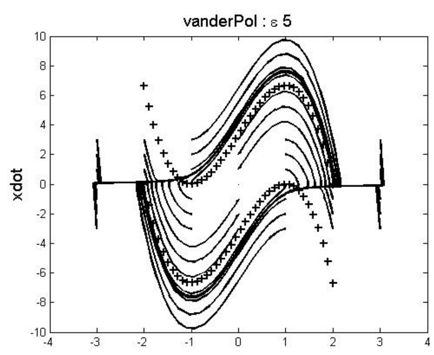
Now look at the phase plane story for the rotating pendulum:
solutions for $\lambda > 1$: five critical points, $-\pi$, 0, and π are saddles and +- 33° are centers. Almost all trajectories are periodic, four different kinds:

1) for sufficiently large velocity, making one revolution after another,
2) oscillating with large amplitude, passing through both stable θ-values (but of course not with velocity zero)
3) two families of small amplitude oscillations, one for each stable critical point.

The different families are separated by special "separatrix" solutions which travel from one unstable critical point to another.

There is yet another kind of solution, the *limit cycle*, a trajectory which approaches a periodic orbit, and the van der Pol model of a vacuum tube oscillator is the most publicized one:

$$\ddot{x} + \varepsilon(1 - x^2)\dot{x} + x = 0, \varepsilon > 0$$



and here is the phase portrait for ε = 5:
One may guess that when ε is large, parts of the limiting periodic orbit must be described by a balance between the highest derivative and the "large" term,

$$\ddot{x} + \varepsilon\dot{x}(1 - x^2) = 0, \dot{x} + \varepsilon(x - x^3/3) = const.$$

Values of const. (+2/3ε and -2/3ε) to make the curve have *x*-axis as horizontal tangent will splice this part together with the other, as we shall see in a second. Plotted as crosses.
The other part should then come from a balance of the *x* term and the xdot-term. With ε large this means xdot small, and that also looks correct.

The Hopf bifurcation example (L6) is easier since it splits into two single equations, and admits an analytical solution which shows the nature of the limit cycle.

The final bit of info. about 2D phase portraits was shown by Ivar Bendixson 1901, after Poincaré discovered it but did not manage a full proof.

The Poincaré-Bendixson theorem
Any orbit of a 2D continuous dynamical system which stays in a close and bounded subset of the phase space forever must either tend to a critical point or to a periodic orbit.

As the Lorenz example showed, no such theorem for >2D systems. It is the special topology of the plane which makes the difference: A closed curve (periodic orbit) divides the plane into two disjoint sets, and orbits cannot cross the boundary (the folklore theorem).