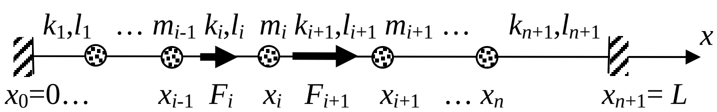
***Lecture 7: Strang's framework for Applied Math.: Graph models***

We have seen the model

$$\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{f} \end{pmatrix} \Rightarrow -\mathbf{A}^T\mathbf{CAx} = \mathbf{f} - \mathbf{A}^T\mathbf{Cb}$$

for mechanical equilibrium, with **C** a positive definite (often diagonal: spring constants) **f** external "forces", **y** the internal "forces" or Lagrange multipliers. The matrix **A** will now be studied in more detail.

## A line of springs




For each mass point $m_i$, Newton's law:

$$m_i\ddot{x}_i = F_{i+1} - F_i + f_i^e, i = 1,...,n \qquad (1)$$

where $f^e$ are the external forces. The springs are assumed linearly elastic (Hookean) so

$$F_i = k_i(l_i - l_i^0) \qquad (2)$$

where $l^0$ is the length of the unloaded spring, and finally, the relation between spring length and the coordinates $x_i$

$$l_1 = x_1; l_i = x_i - x_{i-1}, i = 2,...,n; l_{n+1} = L - x_n \qquad (3)$$

The equations are of very different origin: (1) is a "law of Nature", (3) is an obvious consequence of how we choose to parametrize the model, and (2) captures the physics. The relation between extension and force is an expression of observations, and holds only for limited extension, etc. This is an example of an empirical "constitutive equation" needed to close the system of equations, and it can of course be challenged, refined, etc. But (1) and (3) are not subject to discussion.

Making the obvious vectors out of the $l$, $x$, and $F$,

$$\mathbf{M\ddot{x}} = -\mathbf{A}^T\mathbf{F} + \mathbf{f}^e = 0$$

$$\mathbf{F} = \mathbf{C}(\mathbf{l} - \mathbf{l}^0), \mathbf{C} = diag(k_i) \Rightarrow \mathbf{l} = \mathbf{l}^0 + \mathbf{C}^{-1}\mathbf{F}$$

$$\mathbf{l} = \mathbf{Ax} + \mathbf{b} : \mathbf{C}^{-1}\mathbf{F} - \mathbf{Ax} = \mathbf{b} + \mathbf{l}^0$$

where **A** is the *difference matrix* (shown for $n$ = 4, so there are 5 springs)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ L \end{pmatrix}$$

The block system becomes

$$\begin{pmatrix} \mathbf{C}^{-1} & -\mathbf{A} \\ -\mathbf{A}^T & 0 \end{pmatrix}\begin{pmatrix} \mathbf{F} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{b} + \mathbf{l}^0 \\ \mathbf{f}^e \end{pmatrix}$$

**A** is also the edge-node incidence matrix for the directed graph whose vertices are the mass points and the edges are the springs. The Schur complement system to solve, after we eliminate the forces, has coefficient (= stiffness) matrix $\mathbf{K} = \mathbf{A}^t\mathbf{CA}$. It is symmetric and positive definite, because its columns are independent (easy to see). Here are its elements:

- If an edge, say number $k$, runs between nodes $i$ and $j$ then $K_{ij} = -k_k$.
- All rows but the first and last sum to zero: **A** takes differenes, so **A1** = 0, except in

rows 1 and $n$+1.

- Indeed, $K_{jj}$ = sum of the spring constants of all edges connected to node $j$.
- It is tri-diagonal

$$\mathbf{A}^t\mathbf{CA} = \begin{pmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{pmatrix}$$

- **K** is an $M$-matrix: Its inverse has all positive elements.

*Exercise:* Prove the $M$-matrix property by considering $(\mathbf{I}-\mathbf{B})^{-1} = \sum_{k=0}^{\infty}\mathbf{B}^k$ which is true whenever $\|\mathbf{B}\| < 1$.

### Equilibrium problems described by directed graphs

One may think of an electric circuit as the prototype model in the following: current flows in its branches, driven by electric potential differences between the vertices; Ohm's law for a resistive branch is $V = RI$.

Consider a directed graph with $n$ vertices and $m$ edges, with "flow" $y_i$ in edge $i$, and "potential" $x_k$ at vertex $k$. The graph edge-node incidence matrix is $\mathbf{A}_0$ (the reason for the 0 will be apparent shortly): edge $k$ from node $i$ to node $j$ means $A_{ki} = -1$, $A_{kj} = +1$, the rest zeros. The vector of potential differences $e_j$, $j = 1,\ldots,m$, across the edges is given by

$\mathbf{e} = \mathbf{A}_0\mathbf{x}$

$\mathbf{A}_0\mathbf{1} = 0$ (row sums zero), so $\mathbf{A}_0$ has at most $n$-1 linearly independent columns. Since only potential differences matter, we may assign 0 to an arbitrary $x$, say $x_k$ Then only $n-1$ $x$-values are unknown and we can remove column $k$ from $\mathbf{A}_0$ to produce the reduced incidence matrix **A**, still $\mathbf{e} = \mathbf{Ax}$.

The transpose $\mathbf{A}^T$ acts on the flow variables **y**:

$\mathbf{A}^T\mathbf{y}$ = net flow into the vertices from the branches = external sinks

In other words, *Kirchhoff's current law* is

$\mathbf{A}^T\mathbf{y} = \mathbf{f}$

where **f** is the external "current" source. This is a *conservation law* stating that electric charge is neither created nor destroyed.

The system is closed by the flow vs. potential difference relation for each branch:

$\mathbf{e} = \mathbf{b} - \mathbf{C}^{-1}\mathbf{y}$

allowing for external driving "forces" (think of batteries) **b**. Putting it together:

$\mathbf{Ax}+\mathbf{C}^{-1}\mathbf{y} = \mathbf{b}$

$\mathbf{A}^T\mathbf{y} = \mathbf{f}$

…

### Small strain deformation of trusses

A structure built from linear elements ("sticks"), joined so the joints cannot support any torque, is called a truss (Sv. "Fackverk"). The elements are compressed or extended and respond by forces, which we assume to be Hookean: Force is proportional to extension. "CEINOSTUV" was the anagram, for: *Ut Tensio, Sic Vis*.

Let the vertices have coordinates $\mathbf{X}_j = (x_j,y_j)$ (2D plane) when no loads are present, and $\mathbf{X}_j + \mathbf{x}_j$ under load. The notation which will be used is that the change in distance between nodes $i$ and $j$ under loading is $d_{ij}$, which is small compared to the undeformed length $L_{ij}$ of the bar:

$$d_{ij} = \left\|\mathbf{X}_j + \mathbf{x}_j - (\mathbf{X}_i + \mathbf{x}_i)\right\|_2 - \left\|\mathbf{X}_j - \mathbf{X}_i\right\|_2 = \|\mathbf{Y} - \mathbf{y}\|_2 - \|\mathbf{Y}\|_2 =$$

$$= \sqrt{(\mathbf{Y}+\mathbf{y})^t(\mathbf{Y}+\mathbf{y})} - \sqrt{\mathbf{Y}^T\mathbf{Y}} = \sqrt{\mathbf{Y}^T\mathbf{Y} + 2\mathbf{y}^T\mathbf{Y} + O(\|\mathbf{y}\|^2)} - \sqrt{\mathbf{Y}^T\mathbf{Y}} =$$

$$= \sqrt{\mathbf{Y}^T\mathbf{Y}}\left(1 + \frac{\mathbf{y}^T\mathbf{Y}}{\mathbf{Y}^T\mathbf{Y}}\right) - \sqrt{\mathbf{Y}^T\mathbf{Y}} + O(\|\mathbf{y}\|^2) = \frac{\mathbf{y}^T\mathbf{Y}}{\sqrt{\mathbf{Y}^T\mathbf{Y}}} = (\mathbf{x}_j - \mathbf{x}_i)^T\hat{\mathbf{Y}}_{ij}$$

where the $\mathbf{Y}_{ij}$ is the unit vector $(\cos\theta_{ij}, \sin\theta_{ij})$ along the bar between $i$ and $j$.

The equilibrium is formulated by the Lagrange equations. Let there be external forces $\mathbf{f}^e_j$ on vertex $j$:

$$\min\ W = \sum_{k,m}\frac{1}{2}k_{km}d_{km}^2 - \sum_j \mathbf{x}_j^T\mathbf{f}_j^e,$$

subject to $d_{km} - (\mathbf{x}_k - \mathbf{x}_m)^T\hat{\mathbf{Y}}_{km} = 0, k, m = 1,2,\ldots,n$

$$L = \sum_{k,m}\frac{1}{2}k_{km}d_{km}^2 - \sum_j \mathbf{x}_j^T\mathbf{f}_j^e + \sum_{k,m}\lambda_{km}(d_{km} - (\mathbf{x}_k - \mathbf{x}_m)^T\hat{\mathbf{Y}}_{km})$$

$$\frac{\partial L}{\partial \mathbf{x}_r} = -\mathbf{f}_j^e - \sum_{k,m}\lambda_{km}(\delta_{kr} - \delta_{mr})\hat{\mathbf{Y}}_{km} = 0 \quad (-\mathbf{f}^e = \mathbf{A}^T\boldsymbol{\lambda})$$

$$\frac{\partial L}{\partial d_{rs}} = k_{rs}d_{rs} + \lambda_{rs} = 0 \Rightarrow d_{rs} = -\frac{1}{k_{rs}}\lambda_{rs}$$

$$\frac{\partial L}{\partial \lambda_{rs}} = d_{rs} - \hat{\mathbf{Y}}_{rs}^T(\mathbf{x}_r - \mathbf{x}_s) = 0 : -\frac{1}{k_{rs}}\lambda_{rs} - \hat{\mathbf{Y}}_{rs}^T(\mathbf{x}_r - \mathbf{x}_s) = 0 \quad (\mathbf{C}^{-1}\boldsymbol{\lambda} + \mathbf{A}\mathbf{x} = 0)$$

Most of the possible $n(n-1)/2$ edges do not exist, so the double sums over "$k,m$" are actually a sum over only the $m$ edges. To make that "*Yhat*" formula a little more transparent, write the derivatives for both $x$ and $y$,

$$\frac{\partial L}{\partial x_r} = -f_r^{e,x} - \sum_m \lambda_{rm}\cos\theta_{rm} + \sum_k \lambda_{kr}\cos\theta_{kr} = 0$$

$$\frac{\partial L}{\partial y_r} = -f_r^{e,y} - \sum_m \lambda_{rm}\sin\theta_{rm} + \sum_k \lambda_{kr}\sin\theta_{kr} = 0$$

$$\frac{\partial L}{\partial \lambda_{rs}} = -\frac{1}{k_{rs}}\lambda_{rs} - (x_r - x_s)\cos\theta_{rs} - (y_r - y_s)\sin\theta_{rs} = 0$$

Leaving aside for the moment the "grounding" necessary to make a non-singular system, we let $\mathbf{A}$ be the edge-node incidence matrix for the graph, and replace the +-1's by +-cosθ to produce the $\mathbf{A}^{\cos}$ matrix, and $\mathbf{A}^{\sin}$, analogously. Sorting the unknowns in the order $\boldsymbol{\lambda}$, $\mathbf{x}$, $\mathbf{y}$ gives the matrix:

$$\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{A}^{\cos} & \mathbf{A}^{\sin} \\ \mathbf{A}^{\cos,T} & 0 & 0 \\ \mathbf{A}^{\sin,T} & 0 & 0 \end{pmatrix}\begin{pmatrix}\boldsymbol{\lambda} \\ \mathbf{x} \\ \mathbf{y}\end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f}^{e,x} \\ \mathbf{f}^{e,y}\end{pmatrix}$$

Is this system non-singular? Elimination of $\boldsymbol{\lambda}$ leads to

$$\begin{pmatrix} \mathbf{A}^{\cos,T}C\mathbf{A}^{\cos} & \mathbf{A}^{\cos,T}C\mathbf{A}^{\sin} \\ \mathbf{A}^{\sin,T}C\mathbf{A}^{\cos} & \mathbf{A}^{\sin,T}C\mathbf{A}^{\sin} \end{pmatrix}\begin{pmatrix}\mathbf{x} \\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{f}^{e,x} \\ \mathbf{f}^{e,y}\end{pmatrix}$$

and the diagonal blocks are definite, if the columns of $\mathbf{A}$ are independent. Clearly,

$(\mathbf{1},\mathbf{0})^T$ and $(\mathbf{0},\mathbf{1})^T$ are in the null-space: translation in *x*- and *y*-direction do not deform the truss and create no forces. But also rotation of the graph as a rigid body gives no deformation. What (*x*,*y*) corresponds to (infinitesimal) rigid body rotation ε around, say, the point **P**?

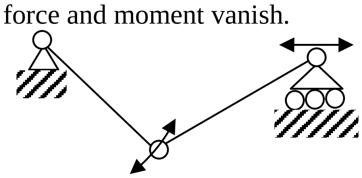
$$\mathbf{x}_j = \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\mathbf{X}_j - \mathbf{P}); \mathbf{x}_k - \mathbf{x}_j = \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\mathbf{X}_k - \mathbf{X}_j) = \varepsilon L_{kj} \begin{pmatrix} -\sin\theta_{kj} \\ \cos\theta_{kj} \end{pmatrix}$$

makes the $(\mathbf{x},\mathbf{y})^T$ orthogonal to all rows of the matrix.

The null space is three-dimensional so at least three constraints must be invoked, by setting selected displacements to zero (“grounding”) and removing the corresponding column(s) of $\mathbf{A}^{\cos}$ and $\mathbf{A}^{\sin}$.

Several cases:
a) Stable, with unique solution, non-singular matrix
1. Statically determinate: the force equation $\mathbf{A}^T\boldsymbol{\lambda} = \mathbf{f}$ can be solved for **λ**.
2. Indeterminate, the “standard” case, must solve for (**x**,**y**).

b) Unstable, singular matrix
1. Rigid body motions allowed: Solutions exist only if the net force and moment vanish.
2. Mechanism: Constraints rule out rigid body motions, still there are deformations which do not change the length of the bars. Example, a slider mechanism, used in reciprocating engines to convert linear motion into rotation.



## Dynamics

The equations are perfectly valid also for a moving truss, when we include the inertial and damping forces, as long as the assumption on *small deformations* of the bars is valid. In particular, there is no assumption of small displacements (**x**,**y**). We obtain

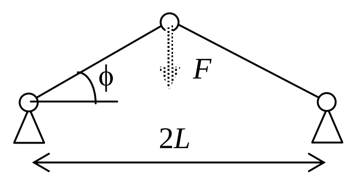
$$\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{A}^{\cos} & \mathbf{A}^{\sin} \\ \mathbf{A}^{\cos,T} & 0 & 0 \\ \mathbf{A}^{\sin,T} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f}^{e,x} + \mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} \\ \mathbf{f}^{e,y} + \mathbf{M}\ddot{\mathbf{y}} + \mathbf{D}\dot{\mathbf{y}} \end{pmatrix}$$

$$\mathbf{A} = \mathbf{A}(\mathbf{X}+\mathbf{x}, \mathbf{Y}+\mathbf{y})$$

where the mass matrix **M** is diag($m_i$) if the masses are point masses at the vertices, and the damping matrix **D**, for a simplistic force proportional to the mass point velocity, is diag($D_i$). The state vector is (**x**,**y**) and (**u**,**v**), the velocities in *x*- and *y*-directions. The algorithm is as follows: From **x** and **y** compute the **A**-matrix, and then the bar forces from the first *m* rows, and the summed *x*- and *y*-forces on the vertices from the last rows; Then compute the accelerations, and take a time-step.

## Equilibrium of non-linear truss: bifurcation, hysteresis

The non-linear truss-model can be used also for static analysis, and it is easy to find examples with non-unique solutions, and bifurcations. Here is a snap-through case: Two bars, both grounded, and a force acting perpendicularly to the line between the supports. Let the unloaded length of the bars be $L_0$, and the spring constant *K*.

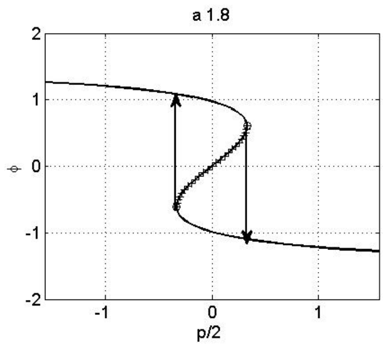



Then, with $p = F/(KL)$, $a = L_0/L > 1$

$$W = (1/\cos\phi - a)^2 + p\tan\phi; W' = 2(1/\cos\phi - a)\frac{\sin\phi}{\cos^2\phi} + p\frac{1}{\cos^2\phi}$$

$$p/2 = a\sin\phi - \tan\phi$$

The graph of the solution, right: Starting at $p$ small, with $\phi$ close to 1, when $p$ increases to 0.33, $\phi$ decreases to 0.606, and then snaps through to –1.1; further increase of $p$ decreases $\phi$ towards $-\pi/2$. If $p$ is *de*creased again, the solution point follows the lower curve until $p$ = –0.33, $\phi$ = –0.606 and then $\phi$ snaps through to +1.1: *Hysteresis*. The part of the solution curve marked 'o' is unstable, as the sign of the second derivative of $W$ tells.




### *Non-linear truss model*

The question arose on the relation of the snap-through geometrically non-linear model to Strang's linear(ized) framework: We did the snap-through in a way to get as quickly as possible to the properties, taking whatever short-cuts made possible by the simplicity.

- The coordinates /degrees of freedom were chosen (the angle) so the constraints necessary to remove rigid body motion are built in. The general truss model presented uses Cartesian coordinates of the joints, and needs explicit enforcement of constraints.
- With only one state variable the "incidence matrix" which forms differences is hard to spot: it is a number.
- We eliminated the "extension" variable immediately so the bar forces never appeared - the "Schur complement" $\mathbf{A}^T\mathbf{CA}$ came out immediately.

So let us redo the model and linearize it after the formulation; in the linear Strang truss model we linearized the relationship between joint position and bar length and used that in the Lagrange equations.

$$W = 2\frac{1}{2}Ke^2 + FL\tan\theta + \lambda\left(e - (\frac{L}{\cos\theta} - L^0)\right)$$

$$\frac{\partial W}{\partial \theta} = \frac{FL}{\cos^2\theta} - \lambda L\frac{\sin\theta}{\cos^2\theta}$$

$$\frac{\partial W}{\partial e} = 2Ke + \lambda, e = -\frac{\lambda}{2K}$$

$$\frac{\partial W}{\partial \lambda} = e - (\frac{L}{\cos\theta} - L^0);$$

$$\begin{cases} \dfrac{\lambda}{2K} + \dfrac{L}{\cos\theta} - L^0 = 0 : C^{-1}\lambda + a(\theta) = 0 \\ \lambda L\dfrac{\sin\theta}{\cos^2\theta} = \dfrac{FL}{\cos^2\theta} : A^T\lambda = f \end{cases}$$

Linearization around a solution $\theta^*$, $\lambda^*$gives

$$\frac{\delta\lambda}{2K} + L\frac{\sin\theta}{\cos^2\theta}\delta\theta = 0, \lambda L\frac{\sin\theta}{\cos^2\theta} = \frac{FL}{\cos^2\theta} :$$

$$\begin{pmatrix} C^{-1} & A(\theta^*) \\ A^T(\theta^*) & \dfrac{\lambda L}{\cos\theta^*} \end{pmatrix}\begin{pmatrix} \delta\lambda \\ \delta\theta \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{L}{\cos^2\theta^*} \end{pmatrix}\cdot\delta F; A(\theta) = L\frac{\sin\theta}{\cos^2\theta}$$

which does fit Strang's framework, with $A$ the derivative of $a$. The replacement of a zero by a non-zero block comes from the repeated differentiation.

### Root following (Homotopy, continuation, ...)

The bifurcation analysis was illustrated by drawing curves (see last lecture notes). The linearized system brings the *implicit function theorem* to mind:

Let $\mathbf{f}(\mathbf{x};p)=0$ be $n$ nonlinear equations in the $n$ variables $x_i$ and $p$ a parameter, and assume that we know a solution x* for $p = p^*$. Let the Jacobian matrix be **J**, $\mathbf{J}(\mathbf{x};p)_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x};p)$. Then, the equations define **x** as a function of $p$ in a neighborhood of $p^*$ if $\mathbf{J}(\mathbf{x}^*,p^*)$ is non-singular.

So we may naively try to follow the solution by solving the system of ODE,

$$\frac{d\mathbf{x}}{dp} = -\mathbf{J}^{-1}(\mathbf{x};p)\frac{\partial \mathbf{f}}{\partial p}; \mathbf{x}(p^*) = \mathbf{x}^*$$

and continue at least as long as the matrix is non-singular. We know where the singular points should be: the “turning points” of the curve where the solution snaps. Hopefully that will come out of this analysis, too:

$$\mathbf{J} = \begin{pmatrix} C^{-1} & A(\theta) \\ A(\theta) & \lambda L/\cos\theta \end{pmatrix} = \begin{pmatrix} 0.5/K & A(\theta) \\ A(\theta) & 2KL(L_0 - L/\cos\theta)/\cos\theta \end{pmatrix}$$

where we used that $(\theta,\lambda)$ is a solution.
***Note***: DO NOT replace λ by the seemingly simpler $F/\sin\theta$, because that (re-) introduces the varying parameter $F$; the expression above contains only the state variable $\theta$ and constants.
**J** is singular when $A^2(\theta) = \frac{L(L_0 - L/\cos\theta)}{\cos\theta} : \cos^3\theta^* = \frac{L}{L_0}$ which agrees with our earlier result.

### Negotiation of turning points?

The naive root-follower will give up on approach to a turning point, or a pitchfork, or a Hopf bifurcation, or, indeed, on any interesting point where things happen quickly. One can make the root-follower negotiate turning points by introducing as independent variable the arclength along the curve traced by $(\mathbf{x},p)$ in $R^n$ x $R$; This gives a differential-algebraic system which can be differentiated again to produce an ODE system.
In the **f**-example,

$$\mathbf{f}_\mathbf{x}\dot{\mathbf{x}} + \mathbf{f}_p\dot{p} = 0 \Rightarrow \mathbf{f}_\mathbf{x}\ddot{\mathbf{x}} + \mathbf{f}_{\mathbf{xx}}\dot{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{f}_p\ddot{p} + \mathbf{f}_{pp}\dot{p}^2 + 2\mathbf{f}_{\mathbf{x}p}\dot{\mathbf{x}}\dot{p} = 0$$

$$\dot{\mathbf{x}}^T\dot{\mathbf{x}} + \dot{p}^2 = 1 \Rightarrow \dot{\mathbf{x}}^T\ddot{\mathbf{x}} + \dot{p}\ddot{p} = 0$$

$$\begin{pmatrix} \mathbf{f}_\mathbf{x} & \mathbf{f}_p \\ \dot{\mathbf{x}} & \dot{p} \end{pmatrix}\begin{pmatrix} \ddot{\mathbf{x}} \\ \ddot{p} \end{pmatrix} = \ldots$$

so: regularity of $\mathbf{f}_\mathbf{x}$ is no longer required! But there are $n$ $n$ x $n$ second derivative matrices involved so for $n > 2$ (1?) we need a symbolic differentiation package to make this practical. Numerical root followers do work in $R^{n+1}$ using arclength, but attack a discretized version of the original problem and do not use second derivatives, except possibly for figuring out what sort of singular point is approaching.