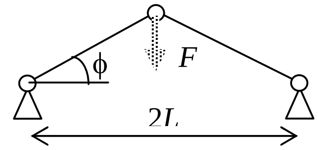
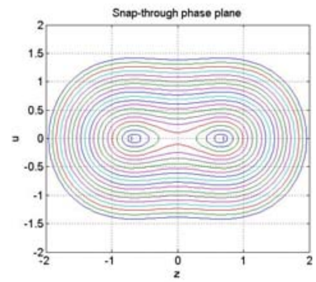
Lecture 7 add: Dynamics of the snap-through model



We have stated repeatedly that the full dynamics system is needed to describe what happens in parametric studies at a bifurcation. The snap-through model, with massless bars and a point mass *m* at the apex, excited by a harmonic load, reads, after choosing *L* for the displacement scale and $T = \sqrt{m/K}$

$$\dot{z} = \dot{u}, \dot{u} = -du - 2z\left(1 - \frac{\alpha}{\sqrt{1+z^2}}\right) + f \sin \omega t$$



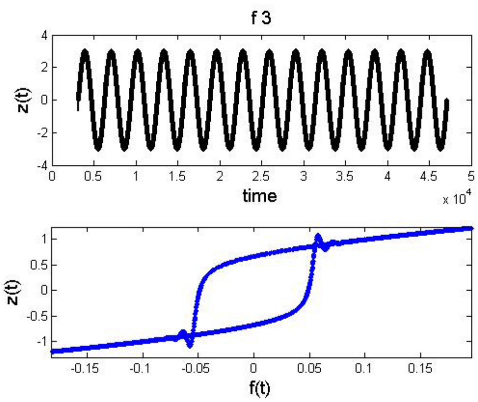
where $\alpha = L_0/L > 1$ and $d > 0$ is the damping coefficient. The (z,u) phase space ($f = 0$) has two stable spiral critical points

$$z^* = \pm\sqrt{\alpha^2 - 1}$$

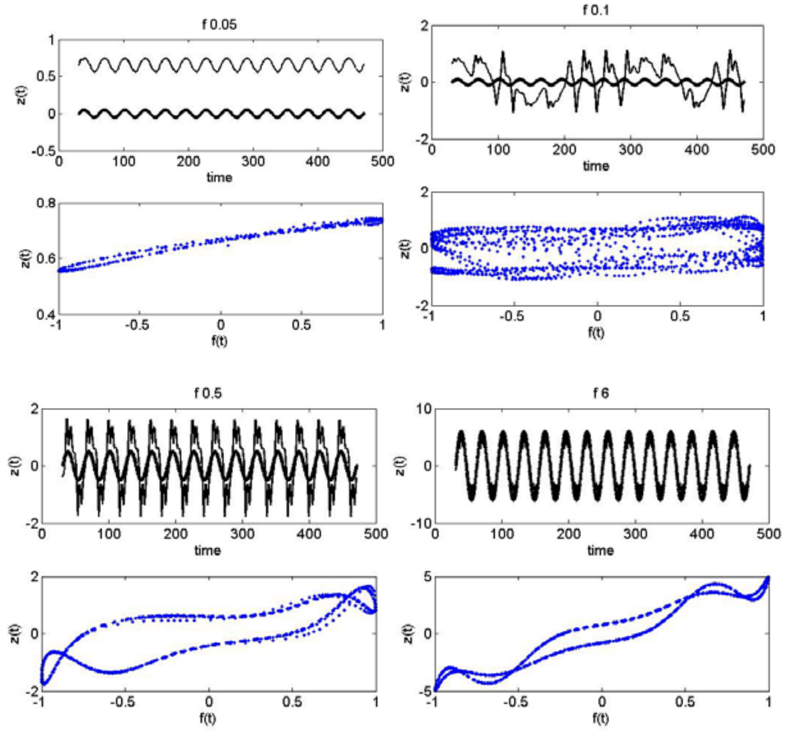
and the origin is a saddle.
Now consider the excited system. Setting $d = 0$, we obtain the total energy as a constant of motion,

$$H = 1/2u^2 + z^2 - 2\alpha\sqrt{1+z^2}$$

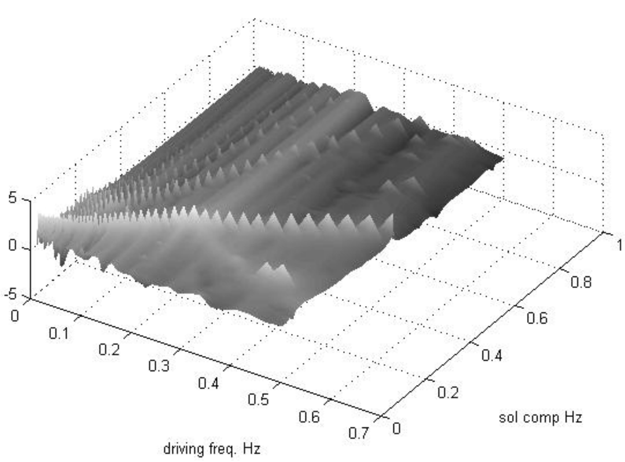
and the iso-lines of *H* are thus solution orbits: It is clear that for small *f* the linearized system will approximate the full model and lead to limit-cycle solutions in the stability basins around either critical point.
Larger forces are more interesting. What is large can be estimated by looking at the maximal force (***Exercise***: Do that!). Smaller forces don't snap-through.
First, we take a very slow driving force so the system is in quasi-equilibrium except at the bifurcations, w = 0.002:



which should be compared to the plot Lect 7. (Same, except for sign change on the force).
Next, what about more rapid excitation? We choose ω as a small multiple of the resonance frequency. Below are trajectories for $\omega = 0.2$ and $f = 0.05$ (small), 0.1 (chaotic?), 0.5 (intermediate) and 6 (large). The small *f* gives linear response; the critical point is selected by the initial data. The *z* vs *f*-map is essentially a line. *f* 0.1 is a border-line case which seems to jump between the basins in a chaotic way, as seen in the *z* vs *f* – plot. The larger *f* show hysteresis; wider hysteresis loop for smaller f. A very large *f* sees the non-linearity of bar forces as small perturbations.



The non-linearity is also noticeable in a spectral analysis. With d = 2 and f = 6, we sweep the driving frequency from 0 to 1/2 “Hz” (1 “Hz” is the resonance frequency of the undamped, small disturbance model).
Any T-periodic function can be decomposed as a sum of sines and cosines of period T, T/2, T/3, … - the Fourier analysis.
For each frequency we compute the solution for 15 periods (like the plots above) and record the amplitude of Fourier components up to 1 “Hz”. The result is shown below. The row of peaks along the 45o line shows that the resonance frequency is dominant. We also see third, fifth, and seventh harmonics. The second is obviously missing as, and indeed all even harmonics. Why?



The sub-harmonic peaks at driving freq. 0.4 – 0.5 are another tell-tale of non-linearity:

$$2\sin^2 x = 1 - \cos 2x$$