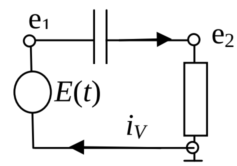
MNA and the differentiator

Homework 4 builds a simulator for passive electrical circuits with capacitances, inductances, and non-linear $i = f(V)$ diodes as well as linear $i = V/R$ resistors. The final formulation (according to the MNA handout and lecture notes L8, with a typo corrected) and distinguishing between linear resistances R (conductance $G = 1/R$) and non-linear (e.g. diodes D) is

$$\begin{bmatrix} A_C C A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt}\begin{pmatrix} e \\ i_L \\ i_V \end{pmatrix} + \begin{bmatrix} A_G G A_G^T & A_L & A_V \\ -A_L^T & 0 & 0 \\ A_V^T & 0 & 0 \end{bmatrix}\begin{pmatrix} e \\ i_L \\ i_V \end{pmatrix} + \begin{pmatrix} A_D f(A_D^T e) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -A_I I \\ 0 \\ E \end{pmatrix}$$

This note was prompted by the failure of ODE15S on the HW4 rectifier model 1b, with a capacitor in parallel with the diode. ODE15S complains that the system may have index > 1 and gives up.

The problem has nothing as such to do with the diode. It is related to the connection of a capacitor direct to a voltage generator, which in fact, as shown below, means that the system has essentially to differentiate $E(t)$ wrt. time t.



So let us study the simplest such arrangement, a circuit known to control engineers as a “differentiator”, for which the MNA equations read

$$A_C = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, A_V = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, A_G = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix};$$

$$C(e_1' - e_2') - i_V = 0$$

$$-C(e_1' - e_2') + (e_2 - 0)/R = 0$$

$$e_1 - 0 = E$$

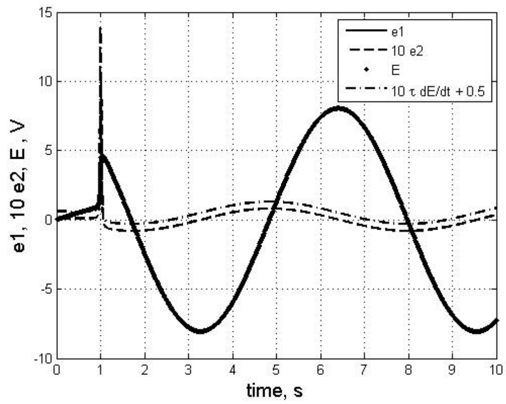
$$\underbrace{\begin{bmatrix} C & -C & 0 \\ -C & C & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M} \frac{d}{dt}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/R & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{K}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}$$

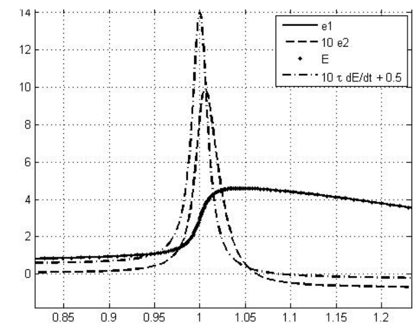
The “mass matrix” M has only rank 1. But K is non-singular, so the matrix M/Δt - K is singular only for possibly a few distinct Δt-values and the solver should encounter no problems in solving for one timestep.

Indeed, a simple test with ODE15S shows this to be true. Here is the solution (left) for C = 10 uF, R = 1 K, $E(t) = \sin t + \left(\frac{2}{\pi}\arctan\left(\frac{t-1}{0.01}\right) + 1\right)\cos t$ with a smoothed Heaviside step-function. Note the jump in E at $t = 1$; $e_1 = E$ of course, so we see only one curve. e_2 is more interesting. The time constant is τ = 10 ms and the driving angular frequency is 1 so we expect to see

e_2 approx. = $\tau\, dE/dt$

which we plot offset by 0.5 so it will not plot exactly on e_2. Note the spike in e_2 at $t = 1$, blown up in the right plot so the τ time constant becomes noticeable. That is the “numerical derivative” of the smoothed Heaviside function. All in all, fair enough.





(Note the different scales on the horizontal axes!).

So what went wrong with ODE15S on the rectifier? First, consider the "Index two" error message. We can turn the system into a proper system of ODE by differentiation (repeated, if necessary) of the algebraic equations.
Once:

$$\underbrace{\begin{bmatrix} C & -C & 0 \\ -C & C & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{M_1} \frac{d}{dt}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1/R & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{K_1}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ dE/dt \end{pmatrix}$$

and the mass matrix now has rank two, at *the price of differentiation of the input signal E(t)*. So proceed by elimination to create a row of zeros in M (always possible when M is rank-deficient) , then differentiate that row:

Elimination :

$$\underbrace{\begin{bmatrix} C & -C & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{M_2} \frac{d}{dt}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1/R & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{K_2}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ dE/dt \end{pmatrix}$$

... and differentiation again :

$$\underbrace{\begin{bmatrix} C & -C & 0 \\ 0 & 1/R & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{M_3} \frac{d}{dt}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{K_3}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ dE/dt \end{pmatrix}$$

and after TWO differentiations there appears a system of ODEs because M3 is non-singular. ODE15S was right in its guess: The original system does have (differentiation) index two. The analysis above shows that the first differentiation creates problems by differentiation of the input signal because we supply only $E(t)$, not dE/dt, as input. So the system has to do numerical differentiation – in particular, to produce the spike in e_2 – which we know to be an ill-posed procedure. A time-step control based on local error estimates will reduce the time-step ad infinitum and give up. Hence, ODE15S tests for index, finds it to be two, and prefers to give up without trying. Maybe better than producing the wrong solution?

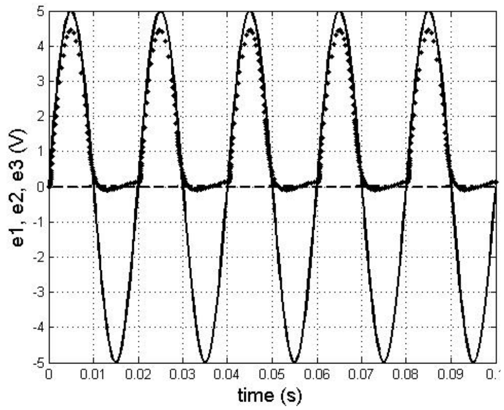
Note:

If we replace the capacitor by a resistance, the mass matrix vanishes and the system has index THREE. Time-stepping is *NOT* the way to solve DC-problems, although BDF1 will produce the exact solution (!). ODE15S cleverly tests for vanishing M and stops.

Solution

Nothing to be done about ODE15S. We must change the mathematical model. It is enough to avoid the "differentiation": add an internal resistance to the generator, so the capacitor is

connected to the resistor – not the generator. It also works to put the resistor in series with the diode capacitance. This was tried on Friday and did not work, but that must be because the resistor was on the wrong side of the capacitor ... Similar steps to be taken with inductances and current generators.
Here is a plot of the rectifier with 100 nF capacitor plus internal resistance 1 mOhm in the generator:



Setting 0 Ohm makes ODE15S break.