Chapter 2: Introduction to Singular Perturbations

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Mathematical Models, Analysis and Simulation, Part I

Introduction

Read: R.E. O'Malley, Jr.: pp 208–223

A mathematical model contains usually many (physical) parameters.

Q: How does the solution depend on the parameters?

- Scale the system appropriately.
- Find out the important parameters.

Observation: Often, one or more of these parameters are very small (or large) in magnitude.

Q: Does the *reduced* model (that is, setting the small parameter to zero) say something about the original problem?

The Problem

- Assume that our problem contains only one small, *positive* parameter ε ($0 < \varepsilon \ll 1$)
- Denote the problem by $\mathbf{P}_\varepsilon$.
- What happens if $\varepsilon \longrightarrow 0$?

A first example

$$\boxed{\mathbf{P}_\varepsilon: \quad f(y,\varepsilon) = y^2 - \varepsilon y - 1 = 0}$$

Solutions:

$$y(\varepsilon) = \frac{1}{2}\left(\varepsilon \pm \sqrt{\varepsilon^2+4}\right)$$

Properties:

- Taylor expansion

$$\boxed{y(\varepsilon) = \pm 1 - \frac{\varepsilon}{2} \pm \frac{\varepsilon^2}{8} + \cdots}$$

- $y(\varepsilon) \longrightarrow \pm 1$ for $\varepsilon \longrightarrow 0$
- ± 1 are the solutions of the limiting (*reduced*) equation $Y_0^2 - 1 = 0$.

A Second Example

$$\boxed{\mathbf{P}_\varepsilon: \quad f(y,\varepsilon) = \varepsilon y^2 + 2y + 1 = 0}$$

- Tempting: Neglect the εy^2 term.
- Problem: $\mathbf{P}_0 : f(y,0) = 2y+1 = 0$ has only one solution.
- **Q:** Where has the second root disappeared?
- **A:** εy^2 cannot be neglected! This term may become large.

Solutions:

$$y^{(1)} = -\frac{1}{2} - \frac{\varepsilon}{8} - \frac{\varepsilon^2}{16} + \cdots,$$
$$y^{(2)} = \boxed{-\frac{2}{\varepsilon}} + \frac{1}{2} + \frac{\varepsilon}{8} + \cdots$$

Observation: $y^{(2)}$ consists of a ”regular” expansion plus a singular correction term.

Singular Perturbations

A perturbation problem $\mathbf{P}_\varepsilon$ is called *regular* if its solution y_ε features smooth dependence on the parameter.

Interpretation: Since ε usually represents a physically meaningful parameter, letting ε tend to 0 corresponds to neglecting the effect of small perturbations.

A perturbation problem is called *singular* if it is not regular.

The first example is a regular perturbation while the second one is a singular perturbation.

Loosely spoken, in a singular perturbation problem, the problem changes its character.

Example: Initial Value Problem

$$\boxed{\mathbf{P}_\varepsilon: \quad \varepsilon\dot{x} + x = 0, \quad x(0) = 1}$$

- Solution: $x(t,\varepsilon) = \exp(-t/\varepsilon)$
- Note: If $\varepsilon < 0$, the solution blows up!
- Limiting solution:

$$x(t,\varepsilon) \longrightarrow \begin{cases} 1, & t = 0, \\ 0, & t > 0. \end{cases}$$

- *The limiting solution does not satisfy the limiting problem*

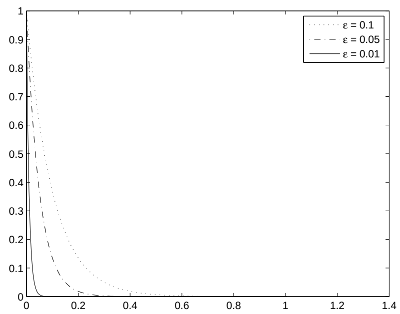
$$X_0 = 0.$$

- A regular expansion of the type

$$x(t,\varepsilon) \sim X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \cdots$$

  cannot hold.
- What has happened to the initial condition?? $\Longrightarrow$ Indication of a singular perturbation problem.

The Example Continued



- The behavior near $t = 0$ is called an *initial layer*.
- The nonuniform convergence takes place in a layer of thickness ε in t.

The initial layer can conveniently be described by introducing the *stretched* variable

$$\boxed{\tau = t/\varepsilon.}$$

In that variable, the problem becomes

$$\frac{dz}{d\tau} + z = 0, \quad z(0) = 1.$$

Linear Initial Value Problems

$$\varepsilon\dot{x} = A(t)x + b(t)$$

Assumptions:

- $A(t)$ is stable for all $t \geq 0$, i.e., all eigenvalue lie in the left complex halfplane.
- A and b are smooth.

Basic idea: Decompose the solution into two parts, a regular one $X(t,\varepsilon)$ and a singular correction term $z(\tau,\varepsilon)$,

$$\boxed{x(t,\varepsilon) = X(t,\varepsilon) + z(t/\varepsilon,\varepsilon)}$$

Notation:

- $X(t,\varepsilon)$ is the outer solution.
- $z(\tau,\varepsilon)$ is the initial layer correction.

The Outer Expansion

$$X(t,\varepsilon) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \cdots$$

Insert this into the differential equation:

$$\varepsilon(\dot{X}_0(t) + \varepsilon\dot{X}_1(t) + \varepsilon^2\dot{X}_2(t) + \cdots) = A(X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \cdots) + b$$

Equating equal powers of ε, we obtain:

$$\begin{aligned}
\varepsilon^0: &\quad 0 = AX_0 + b &&\Rightarrow X_0 = -A^{-1}b\\
\varepsilon^1: &\quad \dot{X}_0 = AX_1 &&\Rightarrow X_1 = -A^{-1}\frac{d}{dt}(A^{-1}b)\\
\varepsilon^2: &\quad \dot{X}_1 = AX_2 &&\Rightarrow X_2 = -A^{-1}\frac{d}{dt}\Big(A^{-1}\big(\frac{d}{dt}(A^{-1}b)\big)\Big)
\end{aligned}$$

This procedure can be continued as long as A is nonsingular and both A and b are sufficiently often differentiable.

Hint: In practice, a few terms will often do.

The Initial Layer Correction

In order to match the initial value $x(0)$, introduce the stretched variable

$$\tau = t/\varepsilon.$$

The initial layer correction becomes

$$z(\tau,\varepsilon) = x(t,\varepsilon) - X(t,\varepsilon)$$

This gives:

$$\begin{aligned}
\frac{d}{d\tau}z(\tau,\varepsilon) &= \varepsilon\frac{d}{dt}z(\tau,\varepsilon)\\
&= \varepsilon\frac{d}{dt}(x(t,\varepsilon) - X(t,\varepsilon))\\
&= (A(t)x(t,\varepsilon) + b(t)) - (A(t)X(t,\varepsilon) + b(t))\\
&= A(\varepsilon\tau)z(\tau,\varepsilon).
\end{aligned}$$

$$\boxed{\frac{d}{d\tau}z(\tau,\varepsilon) = A(\varepsilon\tau)z(\tau,\varepsilon)}$$

Initial value:

$$\boxed{z(0,\varepsilon) = x(0) - X(0,\varepsilon)}$$

The right hand side is known from the outer solution.

Requirement: $z(\tau,\varepsilon) \longrightarrow 0$ for $\tau \longrightarrow \infty$

The Initial Layer Correction (cont)

Formal ansatz:

$$z(\tau,\varepsilon) = z_0(\tau) + \varepsilon z_1(\tau) + \varepsilon^2 z_2(\tau) + \cdots$$

Near $t = 0$, we can use the Taylor expansion (Note: $\dot{} := d/dt$!)

$$A(t) = A(\varepsilon\tau) = A(0) + \varepsilon\tau\dot{A}(0) + \frac{1}{2}\varepsilon^2\tau^2\ddot{A}(0) + \cdots$$

Inserting this into the differential equation, we obtain

$$\frac{d}{d\tau}z(\tau,\varepsilon) = A(\varepsilon\tau)z(\tau,\varepsilon)$$

$$\frac{d}{d\tau}z_0(\tau) + \varepsilon\frac{d}{d\tau}z_1(\tau) + \varepsilon^2\frac{d}{d\tau}z_2(\tau) + \cdots = \Big[A(0) + \varepsilon\tau\dot{A}(0) + \frac{1}{2}\varepsilon^2\tau^2\ddot{A}(0) + \cdots\Big] \times (z_0(\tau) + \varepsilon z_1(\tau) + \varepsilon^2 z_2(\tau) + \cdots)$$

The Initial Layer Correction (cont)

Equating coefficients,

$$\varepsilon^0: \quad \frac{d}{d\tau}z_0 = A(0)z_0 \qquad z_0(0) = x(0) - X_0(0)$$

$$\varepsilon^1: \quad \frac{d}{d\tau}z_1 = A(0)z_1 + \tau\dot{A}(0)z_0 \qquad z_1(0) = -X_1(0)$$

Properties:

- $A(0)$ is a stable matrix. Hence, $z_* = 0$ is a stable equailibrium point.
- *All solutions z_i are decaying exponentially towards 0.*

We have now formally an expansion of $x(t,\varepsilon)$:

$$X(t,\varepsilon) + z(t/\varepsilon,\varepsilon).$$

Q: Will it converge?

A: In generally, not! This is similar to Taylor expansions.

Asymptotic Expansions

Let us consider partial sums,

$$X^N(t,\varepsilon) := \sum_{j=0}^{N}(X_j(t) + z_j(t/\varepsilon))\varepsilon^j$$

The representation is called an *asymtotic expansion* if, for any N, there exist a constant B_N such that

$$\boxed{|x(t,\varepsilon) - X^N(t,\varepsilon)| \le B_N\varepsilon^{N+1},}$$

or, alternatively

$$x(t,\varepsilon) - X^N(t,\varepsilon) = O(\varepsilon^{N+1}).$$

Notation:

$$\boxed{x(t,\varepsilon) \sim X(t,\varepsilon) + z(t/\varepsilon,\varepsilon)}$$

Note:

- Equality does not hold in general!!
- Away from the left boundary, $z(t/\varepsilon,\varepsilon)$ is negligible.

Matched Asymtotic Expansions

In the literature, often a slightly different approach is taken:

1. Find an asymptotic expansion of the outer solution $X(t,\varepsilon)$.
2. Transform the system into stretched variables,

$$\frac{d}{d\tau}v = A(\varepsilon\tau)v + b(\varepsilon\tau), \quad v(0) = x(0).$$

3. Find an asymptotic expansion of the *inner solution* $v(\tau,\varepsilon)$.
4. Apply matching rules with the outer expansion. These matching rules depend on the outer expansion, e.g.,

$$\lim_{\tau\to\infty} v_0(\tau) = \lim_{t\to 0} X_0(t).$$

Nonlinear Problems

- The construction principle is similar to the one given above.
- The construction is technically often very expensive.
- Often, the first few terms will do.
- Example

$$\dot{x} = f(x,y,t,\varepsilon),$$
$$\varepsilon\dot{y} = g(x,y,t,\varepsilon)$$

  with given initial values $x(0)$ and $y(0)$.
  Under the assumption that the reduced system

$$\boxed{\begin{aligned}\dot{X}_0 &= f(X_0,Y_0,t,0),\\ 0 &= g(X_0,Y_0,t,0)\end{aligned}}$$

  has a solution such that $g_y(X_o(t),Y_0(t),t,0)$ is stable, the existence of an asymptotic expansion can be proven.
- In the homework, you will consider a simple nonlinear example from biochemistry.